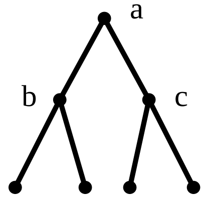
Combinatorial Analysis 21-301: Fall 2003
Homework.
HW7 due Monday 10/27/2003

Q1: A necklace is made of 6 beads strung together in a cycle. How many distinct ways are there of colouring the beads using two colours if the group G is the **dihedral group** generated by rotations about the centre plus flips about a diameter. Using the notation R to represent a clockwise rotation of $\pi/3$ and F to represent a flip on the diameter 1–4, the group G consists of the 12 elements $\{R^iF^j : 0 \leq i \leq 5, j = 1, 2\}$.
Here we can reduce any sequence of rotations and flips to the 12 given by using the relations $R^6 = F^2 = e$ (identity) and $FR = R^5F$.
See http://merganser.math.gvsu.edu/david/reed03/projects/ettingerGuy/ for a nice discussion.
Solution:

g	e	R	R^2	R^3	R^4	R^5	F	RF	R^2F	R^3F	R^4F	R^5F
$\|Fix(g)\|$	64	2	4	8	4	2	16	8	16	8	16	8

So the number of colourings is $\frac{1}{12}(64+2+4+8+4+2+16+8+16+8+16+8) = 13$.

Q2: Consider a complete binary tree on 7 nodes (drawn below): The edges are to be coloured black and blue. How many diferent colourings are there if the group G can rotate the tree below each vertex. This group has 8 elements: Let e_a, e_b, e_c denote rotations under vertices a, b, c respectively. Then dropping the $\circ$ we have $e_ae_b = e_ce_a$, $e_ae_c = e_be_a$ and $e_be_c = e_ce_b$. So G consists of $\{e, e_a, e_b, e_c, e_ae_b, e_ae_c, e_be_c, e_ae_be_c\}$.



Solution:

g	e	e_a	e_b	e_c	e_ae_b	e_ae_c	e_be_c	$e_ae_be_c$
$\lvert Fix(g)\rvert$	64	8	32	32	4	4	16	8

So the number of colourings is $\frac{1}{8}(64 + 8 + 32 + 32 + 4 + 4 + 16 + 8) = 21$.