Combinatorial Analysis 21-301: Fall 2003
Homework.
HW6 due Monday 10/13/2003

Q1: How many sequences $\mathbf{x} = x_1 x_2 \cdots x_n \in \{a, b, c\}^n$ are there for which there is no i such that $x_i x_{i+1} = ab$?
[Hint: The number of k-subsets of $[n-1]$ with no consecutive elements is $\binom{n-k}{k}$. We put down $n-1-k$ markers and then place the k elements into the gaps, including the ends.]
Solution Let

$$A_i = \{\mathbf{x} : \; x_i x_{i+1} = ab\}, \qquad 1 \leq i \leq n-1.$$

The question asks for $\left|\bigcap_{i=1}^{n-1} \bar{A}_i\right|$. To apply inclusion-exclusion we need to find $|A_S|$, for $S \subseteq [n-1]$. Now $A_S = \emptyset$ if S contains a pair of consectitive elements $j, j+1$, otherwise $|A_S| = 3^{n-2|S|}$. So,

$$\left|\bigcap_{i=1}^{n-1} \bar{A}_i\right| = \sum_{k=0}^{n-1} (-1)^k \nu_k 3^{n-2k}$$

where ν_k is the number of k-subsets of $[n-1]$ with no consecutive elements. This number is given in the hint. So the answer to the question is

$$\sum_{k=0}^{n-1} (-1)^k \binom{n-k}{k} 3^{n-2k}$$

Q2: How many symmetric $n \times n$ 0-1 matrices are there in which every row has at least one non-zero?
Solution Let A_i be the set of $n \times n$ 0-1 matrices in which row i and column i are all zero. The question asks for $\left|\bigcap_{i=1}^{n} \bar{A}_i\right|$. To apply inclusion-exclusion we need to find $|A_S|$, for $S \subseteq [n]$. But $|A_S| = 2^{(n-|S|)(n-|S|+1)/2}$ and so the answer to the question is

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} 2^{(n-k)(n-k+1)/2}.$$