Combinatorial Analysis 21-301: Fall 2003
Homework.
HW5 due Monday 10/6/2003

Q1: Let $\mathcal{A} = \{A_1, A_2, \ldots, A_m\}$ be a collection of m distinct subsets of $[n]$ such that if $i \neq j$ then $A_i \cap A_j \neq \emptyset$. Show that $m \leq 2^{n-1}$ and give an example where $m = 2^{n-1}$.
Solution If $m > 2^{n-1}$ then the pigeon-hole principle implies that $\mathcal{A}$ must contain a set X and its complement X^c. This is not possible since $X \cap X^c = \emptyset$. There are two simple examples. In one we divide the set of subsets of $[n]$ into 2^{n-1} pairs of the form X, X^c and choose one set from each pair. Another solution is to take all the sets containing element 1.

Q2: Let $\mathcal{A} = \{A_1, A_2, \ldots, A_m\}$ be a collection of m distinct subsets of $[n]$ such that if $i \neq j$ then (i) $A_i \not\subseteq A_j$, (ii) $A_i \cap A_j \neq \emptyset$ and (iii) $A_i \cup A_j \neq [n]$. Prove that

$$m \leq \binom{n-1}{\lfloor n/2 \rfloor}.$$

[Hint: Show that replacing sets of size greater than $n/2$ leaves a set system in which (i) and (ii) hold. Then apply Theorem 6.6 (Bollobás).]
Solution Suppose that $|A_i| > n/2$. Replace A_i by A_i^c. Now consider any other set A_j. Then $A_i^c \not\subseteq A_j$ for otherwise we would have $A_i \cup A_j = [n]$. Furthermore $A_i^c \cap A_j \neq \emptyset$ for otherwise we would have $A_j \subseteq A_i$. Finally, $A_i^c \cup A_j \neq [n]$ for otherwise we would have $A_i \subseteq A_j$. Thus (i),(ii),(iii) hold. By repeating this we get to the state where all the sets are of size at most $n/2$. We can then apply Theorem 6.6.