Combinatorial Analysis 21-301: Fall 2003
Homework.
HW4 due Monday 9/29/2003

Q1: Show that a simple graph G with n vertices and m edges has at least $\frac{m}{3n}(4m - n^2)$ triangles.

[Hint: An edge (x, y) is in at least $deg(x) + deg(y) - n$ triangles. Sum this over all edges of G.]
Solution From the hint, we see that the number of trinagles in G is at least

$$\frac{1}{3} \sum_{(x,y \in E)} (deg(x) + deg(y) - n) \quad = \quad \frac{1}{3} \left(\sum_{x \in V} deg(x)^2 - mn \right)$$

The 1/3 factor comes from the fact that each triangle is counted 3 times in the sum

$$\geq \quad \frac{1}{3} \left(\frac{4m^2}{n} - mn \right)$$

where we have used the Cauchy-Schwartz inequality.

Q2: Use König's Theorem to prove that every bipartite graph G has a matching of size at least $|E(G)|/\Delta(G)$ where Δ denotes the maximum degree. Use this to show that every subgraph of $K_{n,n}$ with more than $(k-1)n$ edges has a matching of size at least k.
Solution Since each vertex covers at most Δ edges, the size of the minimum cover is at least $|E(G)|/\Delta(G)$. By König's Theorem the size of a maximum matching must be at least this big.
A subgraph of $K_{n,n}$ has maximum degree at most n and so the first part implies there is a matching of size greater than $(k-1)n/n$.