Combinatorial Analysis 21-301: Fall 2003
Homework.
HW3 due Monday 9/15/2003

Q1: The edges of $K_n, n \geq 4$ are coloured Red and Blue in such a way that a Red edge is in at most one triangle. Show that there is a subgraph K_k with $k \geq \lfloor \sqrt{2n} \rfloor$ that contains **no** Red triangles.
[Hint: Choose a maximal set K of k vertices that do not contain a Red triangle. Now use the fact that every vertex not in K makes a Red triangle with an edge contained in K. Also use the fact (to be shown) that if K contains no Red triangles then K must contain at least $k/2$ Blue edges.]
Solution: Let K be as in the hint. We can assume that $k \geq 3$. Since K is maximal, for every $v \notin K$ there is an edge $e_v \subseteq K$ such that v and the endpoints of e_v form the vertices of a Red triangle. Now $v \neq w$ implies that $e_v \neq e_w$. This is because each Red edge is in at most one triangle. We argue next that K contains at least $k/2$ Blue edges. Suppose first that some vertex $v \in K$ is only incident with Red edges. Then, to avoid a Red triangle, all edges not incident with v are Blue and so there are at least $\binom{k-1}{2} \geq k/2$ Blue edges. On the other hand, if every edge is incident with at least one Blue edge, then there are at least $k/2$ Blue edges inside K. Thus

$$\begin{aligned} n - k &\leq \binom{k}{2} - k/2 \\ k^2 &\geq 2n \end{aligned}$$

Q2: Let m be given. Show that if n is large enough then every $n \times n$ 0-1 matrix contains a **principal** $m \times m$ sub-matrix in which all elements below the diagonal are the same, and all elements above the diagonal are the same. (A principal submatrix is one made up of elements from rows $i_1, i_2, \ldots, i_m$ and columns $i_1, i_2, \ldots, i_m$, for some $i_1, i_2, \ldots, i_m$.
[Hint: If A is an $n \times n$ 0-1 matrix, colour K_n with *colour* $A_{i,j}A_{j,i}$ for $i < j$ i.e. with 00 or 01 or 10 or 11.]
Solution: Suppose that $n \geq N(m, m, m, m; 2)$. Then the colouring of K_n induces a mom=nochromatic K_m which in turn gives a principal $m \times m$ sub-matrix M in which all elements below the diagonal are the same, and all

elements above the diagonal are the same. The row and column indices of M are the vertices of the K_m.