Combinatorial Analysis 21-301: Fall 2003.
Answers to HW1

Q1: Let $A_1, A_2, \ldots, A_n$ be n distinct subsets of $[n]$. Show that there is an element $x \in [n]$ such that all of the sets $A_i \setminus \{x\}$ are also distinct.

Proof method: Consider the graph G with **vertices** $A_1, A_2, \ldots, A_n$ and an edge $\{A_i, A_j\}$ with "colour" x whenever $A_i \oplus A_j = \{x\}$. Prove that in any cycle of G, a colour can appear an even number of times. Deduce that one can delete edges of G so that no cycles are left and the number of colours remains the same. Then use the fact that a graph with n vertices and no cycles contains at most $n-1$ edges.

Solution: We want to show that at most $n-1$ colours appear on the edges of G. If we know that x does not appear as a colour, then all of the sets $A_i \setminus \{x\}$ are also distinct. ($A_i \setminus \{x\} = A_j \setminus \{x\}$ iff $A_i = A_j$ (not allowed) or $A_i \oplus A_j = \{x\}$).

Let $C = A_{i_1}, A_{i_2}, \ldots, A_{i_k}, A_{i_1}$ be a cycle in G and suppose that $A_{i_t} \oplus A_{i_{t+1}} = \{x_t\}$ for $t = 1, 2, \ldots k$. Then since $X \oplus X = \emptyset$ for any X, we have

$$\bigoplus_{t=1}^{k} (A_{i_t} \oplus A_{i_{t+1}}) = \emptyset.$$

This implies that

$$\bigoplus_{t=1}^{k} \{x_t\} = \emptyset.$$

So that each "colour" appears an even number of times on the edges of a cycle.

So, removing one edge from a cycle does not reduce the number of edge colours in the graph. Repeating this until there are no cycles, we see that we have the same number of edge colours and a graph with at most $n-1$ edges. (A graph without cycles is a **forest** if it has components $C_1, C_2, \ldots, C_k$, then it has

$$\sum_{i=1}^{k} (|C_i| - 1) = n - k$$

edges.

Q2: Let G be a simple graph with $n > 3$ vertices and no vertex of degree $n-1$. Suppose that for any two vertices of G there is a **unique** vertex joined to both of them. Prove

(i) If x and y are non-adjacent then they have the same degree.

(ii) Now show that G is a **regular** graph i.e. every vertex has the same degree.

[Hint for (i): Suppose that the x, y are non-adjacent. Let the neigbours of x be $x_1, x_2, \ldots, x_k$ and the neighbours of y be $y_1, y_2, \ldots, y_l$, with $x_1 = y_1$. Show that each y_i is adjacent to a unique member of $\{x_1, x_2, \ldots, x_k\}$.]

Solution: Following the hint, let $x_{f(i)}$ be the unique neighbour of x which is also a neighbour of y_i. f is injective because if $f(i) = f(j)$ then y_i and y_j have 2 common neighbours, y and $x_{f(i)}$. So $l \leq k$ and similarly, $k \leq l$.

Now let $G^c = (V, E^c)$ be the **complement** of G i.e. $\{v, w\} \in G^c$ iff $\{v, w\} \notin G$. What we have shown implies that if x, y are joined by a path in G^c then they have the same degree.