Tic Tac Toe and extensions

We consider the following multi-dimensional version of Tic Tac Toe (Noughts and Crosses to the English). The *board* consists of $[n]^d$. A point on the board is therefore a vector $(x_1, x_2, \dots, x_d)$ where $1 \leq x_i \leq n$ for $1 \leq i \leq d$.

A *line* is a set points $(x_j^{(1)}, x_j^{(2)}, \dots, x_j^{(d)})$, $j = 1, 2, \dots, n$ where each sequence $x^{(i)}$ is either (i) of the form $k, k, \dots, k$ for some $k \in [n]$ or is (ii) $1, 2, \dots, n$ or is (iii) $n, n-1, \dots, 1$. Finally, we cannot have Case (i) for all i.

Thus in the (familiar) 3×3 case, the top row is defined by $x^{(1)} = 1, 1, 1$ and $x^{(2)} = 1, 2, 3$ and the diagonal from the bottom left to the top right is defined by $x^{(1)} = 3, 2, 1$ and $x^{(2)} = 1, 2, 3$

Lemma 1. *The number of winning lines in the* (n, d) *game is* $\frac{(n+2)^d - n^d}{2}$.

Proof In the definition of a line there are n choices for k in (i) and then (ii), (iii) make it up to $n+2$. There are d independent choices for each i making $(n+2)^d$. Now delete n^d choices where only Case (i) is used. Then divide by 2 because replacing (ii) by (iii) and vice-versa whenever Case (i) does not hold produces the same set of points (traversing the line in the other direction). $\square$

The game is played by 2 players. The Red player (X player) goes first and colours a point red. Then the Blue player (0 player) colours a different point blue and so on. A player wins if there is a line, all of whose points are that players colour. If neither player wins then the game is a draw. The second player does not have a wnning strategy:

Lemma 2. *Player 1 can always get at least a draw.*

Proof We prove this by considering *strategy stealing*. Suppose that Player 2 did have a winning strategy. Then Player 1 can make an arbitrary first move x_1. player 2 will then move with y_1. Player 1 will now win playing the winning strategy for Player 2 against a first move of y_1. This can be carried out until the strategy calls for move x_1 (if at all). But then Player 1 can make an arbitrary move and continue, since x_1 has already been made. $\square$

0.1 Pairing Strategy

$$\begin{bmatrix} 11 & 1 & 8 & 1 & 12 \\ 6 & 2 & 2 & 9 & 10 \\ 3 & 7 & * & 9 & 3 \\ 6 & 7 & 4 & 4 & 10 \\ 12 & 5 & 8 & 5 & 11 \end{bmatrix}$$

The above array gives a strategy for Player 2 the 5×5 game ($d = 2, n = 5$). For each of the 12 lines there is an associated pair of positions. If Player 1 chooses a position with a number i, then Player 2 responds by choosing the other cell with the number i. This ensures that Player 1 cannot take line i. If Player 1 chooses the * then Player 2 can choose any cell with an unused number. So, later in the game if Player 1 chooses a cell with j and Player 2 already has the other j, then Player 1 can choose an arbitrary cell. Player 2's strategy is to ensure that after all cells have been chosen, he/she will have chosen one of the numbered cells asociatded with each line. This prevents Player 1 from taking a whole line. This is called a *pairing* strategy.

We now generalise the game to the following: We have a family $\mathcal{F} = A_1, A_2, \dots, A_N \subseteq A$. A move consists of one player, taking an uncoloured member of A and giving it his colour. A player wins if one of the sets A_i is completely coloured with his colour.

A pairing strategy is a collection of distinct elements $X = \{x_1, x_2, \ldots, x_{2N-1}, x_{2N}\}$ such that $x_{2i-1}, x_{2i} \in A_i$ for $i \geq 1$. This is called a *draw forcing pairing.* Player 2 responds to Player 1's choice of $x_{2i+\delta}, \delta = 0, 1$ by choosing $x_{2i+3-\delta}$. If Player 1 does not choose from X, then Player 2 can choose any uncoloured element of X. In this way, Player 2 avoids defeat, because at the end of the game Player 2 will have coloured at least one of each of the pairs x_{2i-1}, x_{2i} and so Player 1 cannot have completely coloured A_i for $i = 1, 2, \ldots, N$.

Theorem 1. *If*

$$\left|\bigcup_{A \in \mathcal{G}} A\right| \geq 2|\mathcal{G}| \qquad \forall \mathcal{G} \subseteq \mathcal{F} \tag{1}$$

then there is a draw forcing pairing.

Proof We define a bipartite graph Γ. A will be one side of the bipartition and $B = \{b_1, b_2, \ldots, b_{2N}\}$. Here b_{2i-1} and b_{2i} both represent A_i in the sense that if $a \in A_i$ then there is an edge (a, b_{2i-1}) and an edge (a, b_{2i}). A draw forcing pairing corresponds to a complete matching of B into A and the condition (1) implies that Hall's condition is satisfied. □

Corollary 2. *If $|A_i| \geq n$ for $i = 1, 2, \ldots, n$ and every $x \in A$ is contained in at most $n/2$ sets of $\mathcal{F}$ then there is a draw forcing pairing.*

Proof The degree of $a \in a$ is $\leq 2(n/2)$ in Γ and the degree of each $b \in B$ is at least n. This implies (via Hall's condition) that there is a complete matching of B into A. □

Consider Tic tac Toe when case $d = 2$. If n is even then every array element is in at most 3 lines (one row, one colum and at most one diagonal) and if n is od then every array element is in at most 4 lines (one row, one colum and at most two diagonals). Thus there is a draw forcing pairing if $n \geq 6$, n even and if $n \geq 9$, n odd. (The cases $n = 4, 7$ have been settled as draws. $n = 7$ required the use of a computer to examine all possible strategies.

In general we have

Lemma 3. *If $n \geq 3^d - 1$ and n is odd or if $n \geq 2^d - 1$ and n is even, then there is a draw forcing pairing of (n, d) Tic tac Toe.*

Proof We only have to estimate the number of lines through a fixed point $\mathbf{c} = (c_1, c_2, \ldots, c_d)$. If n is odd then to choose a line L through $\mathbf{c}$ we specify, for each index i whether L is (i) constant on i, (ii) increasing on i or (iii) decreasing on i. This gives 3^d choices. Subtract 1 to avoid the all constant case and divide by 2 becaus each line gets counted twice this way.

When even is even, we observe that once we have chosen in which positions L is constant, L is determined. Suppose $c_1 = x$ and 1 is not a fixed position. Then every other non-fixed position is x or $n - x + 1$. Asuning w.l.o.g. that $x \leq n/2$ we see that $x < n - x = 1$ and the positions with x increase together at the same time as the positions with $n - x + 1$ decrease together. Thus the number of lines through $\mathbf{c}$ in this case is bounded by $\sum_{i=0}^{d-1} \binom{d}{i} = 2^d - 1$. □

0.2 Quasi-probabilistic method

We now prove a theorem of Erdős and Selfridge.

Theorem 3. *If $|A_i| \geq n$ for $i \in [N]$ and $N < 2^{n-1}$, then Player 2 can get a draw in the game defined by $\mathcal{F}$.*

Proof At any point in the game, let C_j denote the set of elements in A which have been coloured with Player j's colour, $j = 1, 2$ and $U = A \setminus C_1 \cup C_2$. Let

$$\Phi = \sum_{i: A_i \cap C_2 = \emptyset} 2^{-|A_i \cap U|}.$$

Suppose that the players choices are $x_1, y_1, x_2, y_2, \ldots,$. Then we observe that immediately after Player 1's first move, $\Phi < N2^{-(n-1)} < 1$.

We will show that Player 2 can keep $\Phi < 1$ through out. Then at the end, when $U = \emptyset$, $\Phi = \sum_{i: A_i \cap C_2 = \emptyset} 1 < 1$ implies that $A_i \cap C_2 \neq \emptyset$ for all $i \in [N]$.

So, now let Φ_j be the value of Φ after the choice of $x_1, y_1, \ldots, x_j$. then if U, C_1, C_2 are defined at precisely this time,

$$\begin{aligned}
\Phi_{j+1} - \Phi_j &= -\sum_{\substack{i: A_i \cap C_2 = \emptyset \\ y_j \in A_i}} 2^{-|A_i \cap U|} + \sum_{\substack{i: A_i \cap C_2 = \emptyset \\ y_j \notin A_i, x_{j+1} \in A_i}} 2^{-|A_i \cap U|} \\
&\leq -\sum_{\substack{i: A_i \cap C_2 = \emptyset \\ y_j \in A_i}} 2^{-|A_i \cap U|} + \sum_{\substack{i: A_i \cap C_2 = \emptyset \\ x_{j+1} \in A_i}} 2^{-|A_i \cap U|}
\end{aligned}$$

We deduce that $\Phi_{j+1} - \Phi_j \leq 0$ if Player 2 chooses y_j to maximise over y, $\displaystyle\sum_{\substack{i: A_i \cap C_2 = \emptyset \\ y \in A_i}} 2^{-|A_i \cap U|}$.

In this way, Player 2 keeps $\Phi < 1$ and obtains a draw. □

In the case of (n, d) Tic Tac Toe, we see that Player 2 can force a draw if (see Lemma 1)

$$\frac{(n+2)^d - n^d}{2} < 2^{n-1}$$

which is implied, for n large, by

$$n \geq (1 + \epsilon) d \log_2 d$$

where $\epsilon > 0$ is a small positive constsnt.