Geography

Start with a chip sitting on a vertex v of a graph or digraph G.

A move consists of moving the chip to a neighbouring vertex. In edge geography, moving the chip from x to y deletes the edge (x, y). In vertex geography, moving the chip from x to y deletes the vertex x.

The problem is given a position (G, v), to determine whether this is a P or N position.

Complexity Both edge and vertex geography are Pspace-hard on digraphs. Edge geography is Pspace-hard on an undirected graph. Only vertex geography on a graph is polynomial time solvable.

1 Undirected Vertex Geography – UVG

Theorem 1. *(G, v) is an N-position in UVG iff every maximum matching of G covers v.*

Proof (i) Suppose that M is a maximum matching of G which covers v. Player 1's strategy is now: Move along M-edge that contains current vertex.

If Player 1 were to lose, then there would exist a sequence of edges $e_1, f_1, \ldots, e_k, f_k$ such that $v \in e_1$, $e_1, e_2, \ldots, e_k \in M$, $f_1, f_2, \ldots, f_k \notin M$ and $f_k = (x, y)$ where y is the current vertex for Player 1 and y is not covered by M. But then if $A = \{e_1, e_2, \ldots, e_k\}$ and $B = \{f_1, f_2, \ldots, f_k\}$ then $(M \setminus A) \cup B$ is a maximum matching (same size as M) which does not cover v, contradiction.

(ii) Suppose now that there is some maximum matching M which does not cover v. Then if (v, w) is Player 1's move, w must be covered by M, else M is not a maximum matching. Player 2's strategy is now: Move along M-edge that contains current vertex. If Player 2 were to lose then there exists $e_1 = (v, w), f_1, \ldots, e_k, f_k, e_{k+1} = (x, y)$ where y is the current vertex for Player 2 and y is not covered by M. But then we have defined an augmenting path from v to y and so M is not a maximum matching, contradiction. □

Note that we can determine whether or not v is covered by all maximum matchings as follows: Find the size σ of the maximum matching G. This can be done in $O(n^3)$ time on an n-vertex graph. Then find the size σ' of a maximum matching in $G - v$. Then v is covered by all maximum matchings of G iff $\sigma = \sigma'$.

2 Undirected Edge Geography – UEG on a bipartite graph

An *even kernel* of G is a non-empty set $S \subseteq V$ such that (i) S is an independent set and (ii) $v \notin S$ implies that $deg_S(v)$ is even, (possibly zero). ($deg_S(v)$ is the number of neighbours of v in S.)

Lemma 1. *If S is an even kernel and $v \in S$ then (G, v) is a P-position in UEG.*

Proof Any move at a vertex in S takes the chip outside S and then Player 2 can immediately put the chip back in S. After a move from $x \in S$ to $y \notin S$, $deg_S(y)$ will become odd and so there is an edge back to S. making this move, makes $deg_S(y)$ even again. Eventually, there will be no $S : \bar{S}$ edges and Player 1 will be stuck in S. □

We now discuss Bipartite UEG i.e. we assume that G is bipartite, G has bipartion consisting of a copy of $[m]$ and a disjoint copy of $[n]$ and edges set E. Now consider the $m \times n$ 0-1 matrix A with $A(i, j) = 1$ iff $(i, j) \in E$.

We can play our game on this matrix: We are either positioned at row i or we are positioned at column j. If say, we are positioned at row i, then we choose a j such that $A(i,j) = 1$ and (i) make $A(i,j) = 0$ and (ii) move the position to column j. An analogous move is taken when we positioned at column j.

Lemma 2. *Suppose the current position is row i. This is a P-position iff row i is in the span of the remaining rows (is the sum (mod 2) of a subset of the other rows). A similar statement can be made if the position is column j.*

Proof Assume the position is row 1 and there exists $I \subseteq [m]$ such that $1 \in I$ and

$$r_1 = \sum_{i \in I \setminus \{1\}} r_i (mod\ 2) \text{ or } \sum_{i \in I} r_i = 0 (mod\ 2) \tag{1}$$

where r_i denotes row i.

I is an even kernel: If $x \notin I$ then either (i) x corresponds to a row and there are no x, I edges or (ii) x corresponds to a column and then $\sum_{i \in I} A(i,x) = 0 (mod\ 2)$ from (1) and then x has an even number of neighbours in I.

Now suppose that (1) does not hold for any I. We show that there exists a ℓ such that $A(1,\ell) = 1$ and putting $A(1,\ell) = 0$ makes column ℓ dependent on the remaining columns. Then we will be in a P-position, by the first part.

Let e_1 be the m-vector with a 1 in row 1 and a 0 everywhere else. Let A^* be obtained by adding e_1 to A as an $(n+1)$th column. Now the row-rank of A^* is the same as the row-rank of A (here we are doing all arithmetic modulo 2). Suppose not, then if r_i^* is the ith row of A^* then there exists a set J such that

$$\sum_{i \in J} r_i = 0 (mod\ 2) \neq \sum_{i \in J} r_i^* (mod\ 2).$$

Now $1 \notin J$ because r_1 is independent of the remaining rows of A, but then $\sum_{i \in J} r_i = 0 (mod\ 2)$ implies $\sum_{i \in J} r_i^* = 0 (mod\ 2)$ since the last column has al zeros, except in row 1.

Thus rank A^* = rank A and so there exists $K \subseteq [n]$ such that

$$e_1 = \sum_{k \in K} c_k (mod\ 2) \text{ or } e_1 + \sum_{k \in K} c_k = 0 (mod\ 2) \tag{2}$$

where c_k denotes column k of A. Thus there exists $\ell \in K$ such that $A(1,\ell) = 1$. Now let $c_j' = c_j$ for $j \neq \ell$ and c_ℓ' be obtained from c_ℓ by putting $A(1,\ell) = 0$ i.e. $c_\ell' = c_\ell + e_1$. But then (2) implies that $\sum_{k \in K} c_k' = 0 (mod\ 2)$. □