Sums of other subtraction games:

In our first example, $g(x) = x \mod 5$ and so for the sum of n such games we have

$$g(x_1, x_2, \ldots, x_n) = (x_1 \mod 5) \oplus (x_2 \mod 5) \oplus \cdots \oplus (x_n \mod 5).$$

Another subtraction game.
One pile:

- A player can remove any even number of chips, but not the whole pile.
- A player can remove the whole pile if it is odd.

The terminal positions are 0 or 2.

Lemma 1. *$g(0) = 0$, $g(2k) = k - 1$ and $g(2k - 1) = k$ for $k \geq 1$.*

Proof 0,2 are terminal postions and so $g(0) = g(2) = 0$. $g(1) = 1$ because the only position one can move to from 1 is 0. We prove the remainder by induction on k.

Assume that $k > 1$.

$$\begin{aligned} g(2k) &= mex\{g(2k-2), g(2k-4), \ldots, g(2)\} \\ &= mex\{k-2, k-3, \ldots, 0\} \\ &= k-1. \\ g(2k-1) &= mex\{g(2k-3), g(2k-5), \ldots, g(1), g(0)\} \\ &= mex\{k-1, k-2, \ldots, 0\} \\ &= k. \end{aligned}$$

□

A more complicated one pile game

Start with N chips. First player can remove up to $N - 1$ chips.

In general, if the previous player took x chips, then the next player can take $y \leq x$ chips.

Thus a games position can be represented by (n, x) where n is the current size of the pile and x is the maximum number of chips that can be removed in this round.

We will use $(n, *)$ to denote a starting position (instead of $(n, n - 1)$).

Lemma 2.

(a) *$(n, *)$ is an N-position if n is not a power of 2.*

(b) *$(2^k, *)$ is a P-position for all $k \geq 1$.*

Proof We prove the lemma by induction on n. One can easily verify it for $n \leq 4$ say.

(a) Suppose that $n = 2^k + x$ where $0 < x < 2^k$. Then A removes x chips and leaves B in position $(2^k, x)$. This is a P-position, since by induction B will lose when allowed to remove up to $2^k - 1 \geq x$ chips.

(b) Assume that the starting position is $(2^{k+1}, *)$. After the first move, the position will be $(2^k + x, 2^k - x)$, assuming that A chooses $2^k - x$ chips.

(i) If $x \leq 0$ then B can win by removing all $2^k + x \leq 2^k - x$ chips.

(ii) Assume $x > 0$. By induction there is a winning strategy for the second player after a move from $(2^k, *)$ to $(x, 2^k - x)$. B can play this strategy and get A to face the position $(2^k, y)$ where $y \leq 2^k - x < 2^k$. By induction, A will lose from this position.

We should check that B will allow this to happen. But in following the winning strategy from $(2^k, *)$ to $(x, 2^k - x)$ to $(0, y)$, B when facing (y', z') will never choose w where $y' > w \geq y'/2$ for otherwise B would lose when A took the remaining $y' - w$ chips. Thus A will never be able to reduce the number of chips below 2^k until B has reduced the number to 2^k. □

Let us next consider a similar game.

Start with N chips. First player can remove up to $N - 1$ chips.

In general, if the previous player took x chips, then the next player can take $y \leq 2x$ chips.

This game has been called Fibonacci Nim: Let $\mathcal{F} = \{1, 2, 3, 5, 8, 13, \ldots F_k, \ldots\}$ be the Fibonacci sequence. Re-call that $F_{k+1} = F_k + F_{k-1}$.

Lemma 3.

(a) $(n, *)$ *is an N-position if* $n \notin \mathcal{F}$.

(b) $(F_k, *)$ *is a P-position for all* $k \geq 1$.

Proof We prove the lemma by induction on n. One can easily verify it for $n \leq 4$ say.

(a) Suppose that $n = F_k + x$ where $0 < x < F_{k-1}$. Player A can employ a winning strategy for a pile of size x whose final move involves y chips, where $y < F_{k-1}/2$; this leaves Player B with a pile of size F_{k-1} from which he/she cannot remove all chips. Player A can always arrange for y to satisfy this property because when faced with a position (y', z'), A will never choose w where $y' > w$ and $w \geq 2(y' - w)$ for otherwise A would lose when B took the remaining $y' - w$ chips.

(b) Assume that the starting position is $(F_{k+1}, *)$. After the first move, the position will be $(F_{k-1} + x, F_k - x)$, assuming that A chooses $F_k - x$ chips.

(i) If $x \leq 0$ then B can win by removing all $F_{k-1} + x \leq 2(F_k - x)$ chips.

(ii) Assume $x > 0$. By induction there is a winning strategy for the second player after a move from $(F_k, *)$ to $(x, F_k - x)$. B can play this strategy and get A to face the position (F_{k-1}, y) where $y \leq F_k - x < 2F_{k-1}$. By induction, A will lose from this position.

We should check that B will allow this to happen. But in following the winning strategy from $(F_k, *)$ to $(x, F_k - x)$ to $(0, y)$, B when facing (y', z') will never choose w where $y' > w \geq y/3$ for otherwise B would lose when A took the remaining $y' - w$ chips. Thus A will never be able to reduce the number of chips below F_{k-1} until B has reduced the number to F_{k-1}. □

The above two results are part of the following general theorem: There are 2 players A and B and A goes first. We have a non-decreasing function f from $N \to N$ where $N = \{1, 2, \ldots\}$ is the set of natural numbers. At the first move A takes any number less than h from the pile, where h is the size of the initial pile. Then on a subsequent move, if a player takes n chips then the next player is constrained to take at most $f(n)$ chips. Thus the above considered the cases $f(n) = n$ and $f(n) = 2n$.

There is a set $\mathcal{H} = \{H_1 = 1 < H_2 < \ldots\}$ of initial pile sizes for which the first player will lose, assuming that the second player plays optimally. Also, if the initial pile size $h \notin \mathcal{H}$ then the first player has a winning strategy.

Theorem 1. *If $f(H_j) \geq H_j$ then $H_{j+1} = H_j + H_\ell$ where*

$$H_\ell = \min_{i \leq j} \{H_i \mid f(H_i) \geq H_j\}.$$

Furthermore, if $f(H_j) < H_j$ then the sequence of losing positions is finite and ends with H_j.

Proof Assume that $f(H_j) \geq H_j$; then $H_\ell = \min_{i \leq j}\{H_i \mid f(H_i) \geq H_j\}$ exists. For any losing position $H_i < H_\ell$, we have $f(H_i) < H_j$, so from an initial pile of size $H_j + H_i$, Player A can remove H_i chips and win, since this leaves B with a pile of size H_j from which he/she cannot remove all chips.

Now let $x < H_\ell$ be a winning position. Given a pile of size $H_j + x$, Player A can employ a winning strategy for a pile of size x whose final move involves y chips, where $f(y) < H_j$; this again leaves Player B with a pile of size H_j from which he/she cannot remove all chips. Player A can always arrange for y to satisfy this property because when faced with a position (y', z'), A will never choose w where $y' > w$ and $f(w) \geq y' - w$ for otherwise A would lose when B took the remaining $y' - w$ chips.

Finally, from a pile of size $H_j + H_\ell$, if Player A takes at least H_ℓ chips then Player B takes the rest and wins. If Player A takes less than H_ℓ then we fall into the preceding paragraph's situation with the roles reversed. This proves the first statement of the theorem.

If $f(H_j) < H_j$, suppose we had $H_{j+1} = H_j + x$ for some $x > 0$. As above, x cannot be any H_i, since then Player A wins from $H_j + H_i$ by removing H_i chips, because $f(H_i) \leq f(H_j) < H_j$. Now since $x < H_{j=1}$, x must be a winning position. Thus Player A can win from $H_j + x$ by employing a winning strategy for x whose final move is y, where $f(y) < H_j$. Thus H_{j+1} is not a losing position – contradiction, i.e. there is no H_{j+1}. □