We now show how to compute the Sprague-Grundy numbering for a sum of games.

For binary integers $a = a_m a_{m-1} \cdots a_1 a_0$ and $b = b_m b_{m-1} \cdots b_1 b_0$ we define $a \oplus b = c_m c_{m-1} \cdots c_1 c_0$ by $c_i = 1$ if $a_i \neq b_i$ and $c_i = 0$ if $a_i = b_i$ for $i = 1, 2, \ldots, m$.

So for example $11 \oplus 5 = 14$.

Theorem 1. *If g_i is the Sprague-Grundy function for game $i = 1, 2, \ldots, n$ then the Sprague-Grundy function g for the sum of the games is defined by*

$$g(x) = g_1(x_1) \oplus g_2(x_2) \oplus \cdots \oplus g_n(x_n)$$

where $x = (x_1, x_2, \ldots, x_n)$.

Proof It is enough to show

1. If $x \in X$ is a sink of D then $g(x) = 0$.
2. If $x \in X$ and $g(x) = b > a \geq 0$ then there exists $x' \in \Gamma^+(x)$ such that $g(x') = a$.
3. If $x \in X$ and $g(x) = b$ and $x' \in \Gamma^+(x)$ then $g(x') \neq g(x)$.

1. If $x = (x_1, x_2, \ldots, x_n)$ is a sink then x_i is a sink of D_i for $i = 1, 2, \ldots, n$. So

$$\begin{aligned} g(x) &= g_1(x_1) \oplus g_2(x_2) \oplus \cdots \oplus g_n(x_n) \\ &= 0 \oplus 0 \oplus \cdots \oplus 0 \\ &= 0. \end{aligned}$$

2. Write $d = a \oplus b$. Then

$$\begin{aligned} a &= d \oplus b \\ &= d \oplus g_1(x_1) \oplus g_2(x_2) \oplus \cdots \oplus g_n(x_n). \end{aligned} \tag{1}$$

Now suppose that we can show there exists i such that $d \oplus g_i(x_i) < g_i(x_i)$. Then since $g_i(x_i) = mex(\Gamma_i^+(x_i))$ there must exist $x_i' \in \Gamma_i^+(x_i)$ such that $g_i(x_i') = d \oplus g_i(x_i)$. Assume without loss of generality that $i = 1$. Then from (1) we have

$$a = g_1(x_1') \oplus g_2(x_2) \oplus \cdots \oplus g_n(x_n) = g(x_1', x_2, \ldots, x_n).$$

Furthermore, $(x_1', x_2, \ldots, x_n) \in \Gamma^+(x)$ and so we will have verified 2.

Let us prove that such an i exists. Suppose that $2^{k-1} \leq d < 2^k$. Then d has a 1 in position k and no higher. Since $d_k = a_k \oplus b_k$ and $a < b$ we must have $a_k = 0$ and $b_k = 1$. Thus there is at least one i such that $g_i(x_i)$ has a 1 in position k. But then $d \oplus g_i(x_i) < g_i(x_i)$ since d "destroys" the kth bit of $g_i(x_i)$ and does not change any higher bit.

3. Suppose without loss of generality that $g(x_1', x_2, \ldots, x_n) = g(x_1, x_2, \ldots, x_n)$ where $x_1' \in \Gamma^+(x_1)$. Then $g_1(x_1') \oplus g_2(x_2) \oplus \cdots \oplus g_n(x_n) = g_1(x_1) \oplus g_2(x_2) \oplus \cdots \oplus g_n(x_n)$ implies that $g_1(x_1') = g_1(x_1)$, contradition. □

If we apply this theorem to the game of Nim then if the position x consists of piles of x_i chips for $i = 1, 2, \ldots, n$ then $g(x) = x_1 \oplus x_2 \oplus \cdots \oplus x_n$.