The positions of a game are partitioned into 2 sets:

- P-positions: The next player cannot win. The previous player can win regardless of the current player's strategy.
- N-positions: The next player has a strategy for winning the game.

The main problem is to determine N and P and what the strategy is for winning from an N-position.

For $x \in X$ let $\Gamma^+(x) = \{y \in X : (x, y) \in A\}$ be the set of pout-neighbours of x.

Labelling procedure

1. Label all sinks with P.
2. Label with N, every position x for which there exists $y \in \Gamma^+(x)$ which is labelled with P.
3. Label with P, every position x for which $\Gamma^+(x)$ is labelled with N.

A position x is an N-position (winning) iff there is a move from x to a losing position for the next player.

The labelling should be carried out in reverse topological order.

Thus there is a unique partition of X into N, P which satisfies the following:

P1 All sinks are in P.

P2 If $x \in N$ then $\Gamma^+(x) \cap P \neq \emptyset$.

P3 If $x \in P$ then $\Gamma^+(x) \subseteq N$.

In Game 1, $P = \{5k : k \geq 0\}$ and in Game 2, $P = \{(x, x) : x \geq 0\}$.

Sprague-Grundy Numbering

For $S \subseteq \{0, 1, 2, \ldots, \}$ let

$$mex(S) = \min\{x \geq 0 : x \notin S\}.$$

Now given an acyclic digraph $D = X, A$ define g recursively by

$$g(x) = \begin{cases} 0 & x \text{ is a sink} \\ mex(\Gamma^+(x)) & \text{otherwise} \end{cases}$$

$g(x)$ can be computed in reverse topological order.

Lemma 1.

$$x \in P \leftrightarrow g(x) = 0.$$

Proof Clearly P1 holds. We check P2 and P3.
P2: If $g(x) > 0$ there must be a $y \in \Gamma^+(x)$ with $g(y) = 0$.
P3: If $g(x) = 0$ there cannot be a $y \in \Gamma^+(x)$ with $g(y) = 0$. □

Sums of games

Suppose that we have n games with digraphs $D_i = (X_i, A_i)$, $i = 1, 2, \ldots, n$. The sum of these games is played as follows. A position is a vector $(x_1, x_2, \ldots, x_n) \in A_1 \times A_2 \times \cdots \times A_n$. To make a move, a player chooses i such that x_i is not a sink of D_i and then replaces x_i by $y \in \Gamma_i^+(x_i)$. The game ends when each x_i is a sink of D_i for $i = 1, 2, \ldots, n$.

Example Nim

In a one pile game, we start with $a \geq 0$ chips and while there is a positive number x of chips, a move consists of deleting $y \leq x$ chips. In this game the N-positions are positive integers and the unique P-position is 0. The Sprague-Grundy numbering is defined by $g(x) = x$.

In general, Nim consists of the sum of n single pile games stsrting with $a_1, a_2, \ldots, a_n > 0$. A move consists of deleting some chips from a non-empty pile.