Combinatorial Games

Game 1 Start with n chips. Players A,B alternately take 1,2,3,4 chips until there are none left. The winner is the person who takes the last chip:

Example

	A	B	A	B	A	
$n = 10$	3	2	4	1		B wins
$n = 11$	1	2	3	4	1	A wins

What is the optimal strategy for this game?

Game 2 Chip placed at point (m, n). Players can move chip to (m', n) or (m, n') where $0 \leq m' < m$ and $0 \leq n' < n$. The player who makes the last move and puts the chip onto $(0, 0)$ wins.

What is the optimal strategy for this game?

Game 3 W is a set of words. A and B alternatel remove words $w_1, w_2, \ldots,$ from W. The rule is that the first letter of w_{i+1} must be the same as the last letter of w_i. The player who makes the last legal move wins.

1 Abstraction

Represent each position by a vertex of a digraph $D = (X, A$. (x, y) is an arc of D iff one can move from position x to position y.

We assume that the digraph is finite and that it is **acyclic** i.e. there are no directed cycles.

The game starts with a chip on vertex x_0 say, and players alternately move the chip to $x_1, x_2, \ldots,$ where $x_{i+1} \in \Gamma^+(x_i)$, the set of out-neighbours of x_i. The game ends when the chip is on a **sink** i.e. a vertex of out-degree zero. The last player to move is the winner.

Example 1: $D = (\{0, 1, \ldots, n\}, A)$ where $(x, y) \in A$ iff $x - y \in \{1, 2, 3, 4\}$.

Example 2: $D = (\{0, 1, \ldots, m\} \times \{0, 1, \ldots, n\}, A)$ where $(x, y) \in \Gamma^+((x', y')))$ iff $x = x'$ and $y > y'$ or $x > x'$ and $y = y'$.

Example 3: $D = (\{(W', w) : W' \subseteq W \setminus \{w\}\}, A)$. w is the last word used and W' is the remaining set of unused words. $(A', w') \in \Gamma^+((A, w))$ iff $w' \in A$ and w' begins with the last letter of w. Also, there is an arc from $(W, \cdot)$ to $(W \setminus \{w\}, w)$ for all w, corresponding to the games start.

We will first argue that such a game must eventually end. A **topological numbering** of digraph D is a map $f : X \to [N]$, $N = |X|$ which satisfies $(x, y) \in A$ implies $f(x) < f(y)$.

Theorem 1. *A finite digraph $D = (X, A)$ is acyclic iff it admits at least one topological numbering.*

Proof Suppose first that D has a topological numbering. We show that it is acyclic. Suppose that $C = (x_1, x_2, \ldots, x_k, x_1)$ is a directed cycle. Then $f(x_1) < f(x_2) < \cdots < f(x_k) < f(x_1)$, contradiction.

Suppose now that D is acyclic. We first argue that D has at least one sink. Thus let $P = (x_1, x_2, \ldots, x_k)$ be a longest simple path in D. We claim that x_k is a sink. If D contains an arc (x_k, y) then either $y = x_i, 1 \leq i \leq k-1$ and this means that D contains the cycle $x_i, x_{i+1}, \ldots, x_k, x_i)$, contradiction or $y \notin \{x_1, x_2, \ldots, x_k\}$ and then (P, y) is a longer simple path than P, contradiction.

We can now prove by induction on N that there is at least one topological numbering. Ifg $N = 1$ and $X = \{x\}$ then $f(x) = 1$ defines a topological numbering.

Now asssume that $N > 1$. Let z be a sink of D and define $f(z) = N$. The digraph $D' = D - z$ is acyclic and by the induction hypothesis it admits a topological numbering, $f : X \setminus \{z\} \to [N-1]$. The function we have defined on X is a topological numbering. If $(x, y) \in A$ then either $x, y \neq z$ and then $f(x) < f(y)$ by our assumption on f, or $y = z$ and then $f(x) < N = f(z)$ ($x \neq z$ because z is a sink). $\square$

The fact that D has a topological numbering implies that the game must end. Each move increases the f value of the current position by at least one and so after at most N moves a sink must be reached.