The **girth** of a graph is the length of the shortest cycle in G. At first sight it seems that a graph having large girth would necessarily be colourable with few colours i.e. there might exist a function f such that if G has girth at least a then its chromatic number would be at most $f(a)$. This is ruled out by the following theorem of Erdős:

Theorem 1. *Let $a, b > 0$ be positive integers. Then there exists a graph with girth $g \geq a$ and chromatic number $\chi \geq b$.*

Proof This follows from the following lemma:

Lemma 1. *Let d be large enough so that $\frac{d}{3\log d} \geq b$. Then there exists a graph G with $n \geq 100d^a$ such that*

(i) *There are at most $2d^a$ cycles of length $\leq a$.*

(ii) *There is no independent set of vertices of size $\geq \frac{2\log d}{d} n$.*

From the lemma, let G' be obtained from G by deleting one vertex from each cycle of length $\leq a$. Then G' has girth $\geq a$ and

$$\chi \geq \frac{n - 2d^a}{\frac{2\log d}{d} n} \geq \frac{d}{3\log d} \geq b.$$

□

Proof of Lemma: Let $p = d/n$ and consider $G_{n,p}$.

Markov Inequality: If X is a non-negative random variable then

$$\mathbf{Pr}(X \geq t) \leq \frac{\mathbf{E}(X)}{t} \qquad \text{for any } t > 0.$$

Proof:

$$\begin{aligned} \mathbf{E}(X) &= \mathbf{E}(X \mid X \geq t)\mathbf{Pr}(X \geq t) + \mathbf{E}(X \mid X < t)\mathbf{Pr}(X < t) \\ &\geq \mathbf{E}(X \mid X \geq t)\mathbf{Pr}(X \geq t) \\ &\geq t\mathbf{Pr}(X \geq t). \end{aligned}$$

□

(i) Let X be the number of cycles of length at most a. Then

$$\begin{aligned} \mathbf{E}(X) &\leq \sum_{k=3}^{a} \binom{n}{k} \frac{(k-1)!}{2} p^k \\ &\leq \sum_{k=3}^{a} \frac{n^k}{k!} \frac{(k-1)!}{2} \frac{d^k}{n^k} \\ &= \sum_{k=3}^{a} \frac{d^k}{2k} \\ &< d^a. \end{aligned}$$

Applying the Markov inequality we get

$$\mathbf{Pr}(X \geq 2d^a) \leq \frac{E(X)}{2d^a} \leq \frac{1}{2}.$$

(ii) Let $k = \left\lceil \frac{2\log d}{d} n \right\rceil$ and Y be the number of independent sets of size k in $G_{n,p}$. Then

$$\begin{aligned}
\mathbf{E}(X) &= \binom{n}{k}\left(1-\frac{d}{n}\right)^{\binom{k}{2}} \\
&\leq \left(\frac{ne}{k}\cdot\left(1-\frac{d}{n}\right)^{\frac{k-1}{2}}\right)^{k} \\
&\leq \left(\frac{de}{2\log d}\cdot e^{d/2n}\cdot e^{-dk/2n}\right)^{k} \\
&\leq \left(\frac{de}{2\log d}\cdot(1+o(1))\cdot\frac{1}{d}\right)^{k} \\
&\to 0.
\end{aligned}$$

So

$$\mathbf{Pr}(G_{n,p} \text{ satisfies (i) and (ii)}) \geq \frac{1}{2} - o(1) > 0.$$

□