We now consider the connectivity of a random graph:

Theorem 1. *Let $\epsilon > 0$ be a constant.*

(a) If $p \geq \frac{(1+\epsilon)\ln n}{n}$ then **whp** *$G_{n,p}$ is connected.*

(b) If $p \leq \frac{(1-\epsilon)\ln n}{n}$ then **whp** *$G_{n,p}$ is noot connected.*

Proof
(a) If $G_{n,p}$ is not connected then there exists a component S of size $1 \leq s \leq n/2$. Now if S is a component then

(i) There is a spanning tree of S in $G_{n,p}$.

(ii) There are no $S : \bar{S}$ edges.

Let $\mathcal{E}_S$ be the event that S is a component. Then

$$\begin{aligned}
\mathbf{Pr}(G_{n,p} \text{ is not connected}) &= \mathbf{Pr}\left(\bigcup_{\substack{S\subseteq[n] \\ 1\leq|S|\leq n/2}} \mathcal{E}_S\right) \\
&\leq \sum_{\substack{S\subseteq[n] \\ 1\leq|S|\leq n/2}} \mathbf{Pr}(\mathcal{E}_S) \\
&= \sum_{s=1}^{n/2}\binom{n}{s}\mathbf{Pr}(\mathcal{E}_{[s]}) \\
&= \sum_{s=1}^{n/2}\binom{n}{s}\mathbf{Pr}(\text{(i) and (ii)}) \\
&= \sum_{s=1}^{n/2}\binom{n}{s}\mathbf{Pr}(\text{(i)})\mathbf{Pr}(\text{(ii)})
\end{aligned}$$

Since (i) and (ii) are independent events

$$= \sum_{s=1}^{n/2}\binom{n}{s}\mathbf{Pr}(\text{(i)})(1-p)^{s(n-s)} \qquad (1)$$

Before we can proceed, we must estimate $\mathbf{Pr}((\text{i}))$.

First Moment Method

Lemma 1. *Let X be a random variable taking values in $\{0, 1, 2 \ldots, \}$. Then*

$$\mathbf{Pr}(X \neq 0) \leq \mathbf{E}(X).$$

Proof Let $p_i = \mathbf{Pr}(X = i)$ for $i = 0, 1, 2 \ldots$. Then

$$\begin{aligned}
\mathbf{Pr}(X \neq 0) &= p_1 + p_2 + \cdots + p_k + \cdots \\
\mathbf{E}(X) &= p_1 + 2p_2 + \cdots + kp_k + \cdots
\end{aligned}$$

□

We apply this in estimating $\mathbf{Pr}((\text{i}))$ as follows: Let X denote the number of spanning trees of S in $G_{n,p}$. Then (i) occurs iff $X \neq 0$. To apply Lemma 1 we need to compute $\mathbf{E}(X)$. Let $m = s^{s-2}$ and let $T_1, T_2, \ldots, T_m$ be an enumeration of the spanning trees of K_s. Let $X_i = 1_{T_i \subseteq G_{n,p}}$. Then $X = X_1 + \cdots + X_m$ and so

$$\mathbf{E}(X) = \mathbf{E}(X_1 + \cdots + X_m) = m\mathbf{Pr}(T_1 \subseteq G_{n,p}) = s^{s-2}p^{s-1}.$$

Thus

$$\begin{aligned}
\mathbf{Pr}(G_{n,p} \text{ is not connected}) &\leq \sum_{s=1}^{n/2} \binom{n}{s} s^{s-2}p^{s-1}(1-p)^{s(n-s)} \\
&\leq \sum_{s=1}^{n/2} \frac{1}{s^2 p} \left(\frac{ne}{s} \cdot sp \cdot (1-p)^{n-s}\right)^s \qquad (2) \\
&\leq nee^{-(n-1)p} + \sum_{s=2}^{n/2} \frac{1}{s^2 p} \left(\frac{ne}{s} \cdot sp \cdot e^{-np/2}\right)^s \\
&\leq ne^{1+p}n^{-(1+\epsilon)} + \frac{n}{\log n} \sum_{s=2}^{\infty} \left(\frac{e(1+\epsilon)\log n}{n^{(1+\epsilon)/2}}\right)^s \\
&= e^{1+p}n^{-\epsilon} + \frac{n}{\log n}\frac{x^2}{1-x}
\end{aligned}$$

where $x = \frac{e(1+\epsilon)\log n}{n^{(1+\epsilon)/2}}$

$$= O(n^{-\epsilon}).$$

We need to verify the inequality

$$\binom{n}{s} \leq \left(\frac{ne}{s}\right)^s, \qquad s \geq 1,$$

which was used in (2).

We do this by induction on s. It is trivial for $s = 1$, because $\binom{n}{1} = n < \frac{ne}{1}$.

So assume it is true for $s \geq 1$. Then

$$\begin{aligned}
\binom{n}{s+1} &= \frac{n-s}{s+1}\binom{n}{s} \\
&\leq \frac{n}{s+1}\left(\frac{ne}{s}\right)^s \\
&= \left(\frac{ne}{s+1}\right)^{s+1} e^{-1}\left(1+\frac{1}{s}\right)^s \\
&\leq \left(\frac{ne}{s+1}\right)^{s+1}
\end{aligned}$$

after using $(1+s^{-1})^s \leq (e^{1/s})^s = e$. □

(b) Now let $p = \frac{(1-\epsilon)\log n}{n}$ and let X be the number of isolated vertices in $G_{n,p}$. Then

$$\begin{aligned}
\mathbf{Pr}(G_{n,p} \text{ is not connected}) &\geq \mathbf{Pr}(X \neq 0) \\
&\geq \frac{\mathbf{E}(X)^2}{\mathbf{E}(X^2)}. \qquad (3)
\end{aligned}$$

We will verify (3) later.

$$\begin{aligned}\mathbf{E}(X) &= n(1-p)^{n-1}\\ &\geq ne^{-(n-1)(p+p^2)}\end{aligned}$$

since $1-p=e^{\log(1-p)}$ and $\log(1-p)=-p-\frac{p^2}{2}-\cdots-\frac{p^k}{k}\geq -p-p^2$

$$\begin{aligned}&\geq ne^{-(n-1)p^2}n^{-(1-\epsilon)}\\ &\rightarrow \infty.\end{aligned}$$

Now let $X_i = 1_{i \text{ is isolated}}$ so that $X = X_1+\cdots+X_n$. Then

$$\begin{aligned}\mathbf{E}(X^2) &= \sum_{i=1}^n \mathbf{E}(X_i^2)+\sum_{i\neq j}\mathbf{E}(X_iX_j)\\ &= \sum_{i=1}^n \mathbf{E}(X_i)+\sum_{i\neq j}\mathbf{E}(X_iX_j)\\ &= \mathbf{E}(X)+n(n-1)(1-p)^{2n-3}\\ &= \mathbf{E}(X)+\frac{n-1}{n(1-p)}\mathbf{E}(X)^2.\end{aligned}$$

So

$$\begin{aligned}\frac{\mathbf{E}(X)^2}{\mathbf{E}(X^2)} &= \frac{\mathbf{E}(X)^2}{\mathbf{E}(X)+\frac{n-1}{n(1-p)}\mathbf{E}(X)^2}\\ &= \frac{1}{\mathbf{E}(X)^{-1}+\frac{n-1}{n(1-p)}}\\ &\rightarrow 1.\end{aligned}$$

It only remains to prove (3). Let $Y = 1_{X\neq 0}$ so that $X = XY$ and $Y^2 = Y$. Now the Cauchy-Schwartz inequality says

$$\mathbf{E}(XY)^2 \leq \mathbf{E}(X^2)\mathbf{E}(Y^2).$$

Now use $\mathbf{E}(XY) = \mathbf{E}(X)$ and $\mathbf{E}(Y^2) = \mathbf{E}(Y) = \mathbf{Pr}(X \neq 0)$.

We can prove the Cauchy-Schwartz inequality as follows: Observe that the quadratic $\mathbf{E}((X+\lambda Y)^2 = \mathbf{E}(X^2)+2\mathbf{E}(XY)\lambda+\mathbf{E}(Y^2)\geq 0$ for all λ. This implies the inequality! □