Random Graphs

The random graph $G_{n,p}$ is obtained by **independently** including each edge of K_n with probability p and excluding it with probability $1-p$.
(Note here that p can be a function of n.)

If H is a spanning subgraph of K_n (i.e. $V(H)=[n]$) with m edges then

$$\mathbf{Pr}(G_{n,p}=H)=p^m(1-p)^{\binom{n}{2}-m}.$$

Suppose that $p=1/2$. Then for every H,

$$\mathbf{Pr}(G_{n,1/2}=H)=\left(\frac{1}{2}\right)^m\left(\frac{1}{2}\right)^{\binom{n}{2}-m}=\left(\frac{1}{2}\right)^{\binom{n}{2}}.$$

So $G_{n,1/2}$ is equally likely to be any H i.e. it is a random spanning sub-graph of K_n.

We study properties that occur *with high probabity* **whp** i.e. with probability $\to 1$ as $n\to\infty$.

The diameter of a graph $G=(V,E)$ is the maximum over pairs $v,w\in V$ of the shortest distance from v to w.

Theorem 1. *If* $p\geq\sqrt{\frac{3\ln n}{n-2}}$ *then* **whp** *the diameter of* $G_{n,p}$ *is at most 2.*

Proof Let

$$\mathcal{E}_{i,j}=\{\not\exists k:\ (i,k)\text{ and }(k,j)\in E(G_{n,p})\}.$$

Let

$$\mathcal{E}=\bigcup_{1\leq i<j\leq n}\mathcal{E}_{i,j}.$$

If $\mathcal{E}$ does not occur then the diameter of $G_{n,p}$ is at most 2.

$$\begin{aligned}
\mathbf{Pr}(\mathcal{E}) &= \mathbf{Pr}\left(\bigcup_{1\leq i<j\leq n}\mathcal{E}_{i,j}\right)\\
&\leq \sum_{1\leq i<j\leq n}\mathbf{Pr}(\mathcal{E}_{i,j})\\
&\leq \sum_{1\leq i<j\leq n}(1-p^2)^{n-2}\\
&\leq \sum_{1\leq i<j\leq n}e^{-(n-2)p^2} \qquad (1)\\
&\leq \sum_{1\leq i<j\leq n}e^{-3\ln n}\\
&= \binom{n}{2}n^{-3}\\
&\to 0.
\end{aligned}$$

(1) follows from the following:

Lemma 1.

$$1+x\leq e^x \textit{ for all } x.$$

Proof

1. $x \geq 0$: $1 + x \leq 1 + x + x^2/2! + x^3/3! + \cdots = e^x$.
2. $x < -1$: $1 + x < 0 \leq e^x$.
3. $x = -y, 0 \leq y \leq 1$: $1 - y \leq 1 - y + (y^2/2! - y^3/3!) + (y^4/4! - y^5/5!) + \cdots = e^{-y}$.

$\square$

We now consider the connectivity of a random graph:

Theorem 2. *Let $\epsilon > 0$ be a constant.*

(a) If $p \geq \frac{(1+\epsilon)\ln n}{n}$ then **whp** *$G_{n,p}$ is connected.*

(b) If $p \leq \frac{(1-\epsilon)\ln n}{n}$ then **whp** *$G_{n,p}$ is noot connected.*

Proof

(a) If $G_{n,p}$ is not connected then there exists a component S of size $1 \leq s \leq n/2$. Now if S is a component then

(i) There is a spanning tree of S in $G_{n,p}$.

(ii) There are no $S : \bar{S}$ edges.

Let $\mathcal{E}_S$ be the event that S is a component. Then

$$\begin{aligned} \mathbf{Pr}(G_{n,p} \text{ is not connected}) &= \mathbf{Pr}\left(\bigcup_{\substack{S\subseteq[n] \\ 1\leq|S|\leq n/2}} \mathcal{E}_S\right) \\ &\leq \sum_{\substack{S\subseteq[n] \\ 1\leq|S|\leq n/2}} \mathbf{Pr}(\mathcal{E}_S) \\ &= \sum_{s=1}^{n/2} \binom{n}{s} \mathbf{Pr}(\mathcal{E}_{[s]}) \\ &= \sum_{s=1}^{n/2} \binom{n}{s} \mathbf{Pr}(\text{(i) and (ii)}) \\ &= \sum_{s=1}^{n/2} \binom{n}{s} \mathbf{Pr}(\text{(i)})\mathbf{Pr}(\text{(ii)}) \end{aligned}$$

Since (i) and (ii) are independent events

$$= \sum_{s=1}^{n/2} \binom{n}{s} \mathbf{Pr}(\text{(i)})(1-p)^{s(n-s)} \tag{2}$$

Before we can proceed, we must estimate $\mathbf{Pr}(\text{(i)})$.

First Moment Method

Lemma 2. *Let X be a random variable taking values in $\{0, 1, 2 \ldots, \}$. Then*

$$\mathbf{Pr}(X \neq 0) \leq \mathbf{E}(X).$$

Proof Let $p_i = \mathbf{Pr}(X = i)$ for $i = 0, 1, 2 \ldots$. Then

$$\begin{aligned} \mathbf{Pr}(X \neq 0) &= p_1 + p_2 + \cdots + p_k + \cdots \\ \mathbf{E}(X) &= p_1 + 2p_2 + \cdots + kp_k + \cdots \end{aligned}$$

□

We apply this in estimating $\mathbf{Pr}$((i)) as follows: Let X denote the number of spanning trees of S in $G_{n,p}$. Then (i) occurs iff $X \neq 0$. To apply Lemma 2 we need to compute $\mathbf{E}(X)$.