Class 23

The probabilistic method

Example 1

Theorem 1. *Assume that* $k \geq 3$. *Then*

$$R(k,k) \geq 2^{k/2}.$$

Proof We must prove that if $n \leq 2^{k/2}$ then there exists a Red-Blue colouring of the edges of K_n which contains no Red k-clique and no Blue k-clique. We can assume $k \geq 4$ since we know $R(3,3) = 6$.

We show that this is true with positive probability in a *random* Red-Blue colouring. So let Ω be the set of all Red-Blue edge colourings of K_n with uniform distribution. Equivalently we independently colour each edge Red with probability 1/2 and Blue with probability 1/2.

Let
$\mathcal{E}_R$ be the event: {There is a Red k-clique} and
$\mathcal{E}_B$ be the event: {There is a Blue k-clique}.

We show

$$\mathbf{Pr}(\mathcal{E}_R \cup \mathcal{E}_B) < 1.$$

Let $C_1, C_2, \ldots, C_N$, $N = \binom{n}{k}$ be the vertices of the N k-cliques of K_n.
Let $\mathcal{E}_{R,j}$ be the event: $\{C_j$ is Red$\}$.
Now

$$\begin{aligned}
\mathbf{Pr}(\mathcal{E}_R \cup \mathcal{E}_B) &\leq \mathbf{Pr}(\mathcal{E}_R) + \mathbf{Pr}(\mathcal{E}_B) \\
&= 2\mathbf{Pr}(\mathcal{E}_R) \\
&= 2\mathbf{Pr}\left(\bigcup_{j=1}^{N} \mathcal{E}_{R,j}\right) \\
&\leq 2\sum_{j=1}^{N} \mathbf{Pr}(\mathcal{E}_{R,j}) \\
&= 2\sum_{j=1}^{N} \left(\frac{1}{2}\right)^{\binom{k}{2}} \\
&= 2\binom{n}{k}\left(\frac{1}{2}\right)^{\binom{k}{2}} \\
&\leq 2\frac{n^k}{k!}\left(\frac{1}{2}\right)^{\binom{k}{2}} \\
&\leq 2\frac{2^{k^2/2}}{k!}\left(\frac{1}{2}\right)^{\binom{k}{2}} \\
&= \frac{2^{1+k/2}}{k!} \\
&< 1.
\end{aligned}$$

□

The result here may be strengthened slightly to state that

$$2\binom{n}{k}\left(\frac{1}{2}\right)^{\binom{k}{2}} < 1 \text{ implies } R(k,k) > n.$$

Example 2 Colouring Problem

Theorem Let $A_1, A_2, \ldots, A_n$ be subsets of A and $|A_i| = k$ for $1 \leq i \leq n$. If $n < 2^{k-1}$ then there exists a partition $A = R \cup B$ such that

$$A_i \cap R \neq \emptyset \text{ and } A_i \cap B \neq \emptyset \qquad 1 \leq i \leq n.$$

[$R =$ Red elements and $B =$ Blue elements.]

Proof Randomly colour A.
$\Omega = \{R, B\}^A = \{f : A \to \{R, B\}\}$, uniform distribution.

$$\mathcal{E} = \{\exists i : \ A_i \subseteq R \text{ or } A_i \subseteq B\}.$$

Claim: $\mathbf{P}(\mathcal{E}) < 1$.

Thus $\Omega \setminus \mathcal{E} \neq \emptyset$ and this proves the theorem.

$$\mathcal{E}_i = \{A_i \subseteq R \text{ or } A_i \subseteq B\}$$

$$\mathcal{E} = \bigcup_{i=1}^{n} \mathcal{E}_i.$$

$$\begin{aligned} \mathbf{P}(\mathcal{E}) &\leq \sum_{i=1}^{n} \mathbf{P}(\mathcal{E}_i) \\ &= \sum_{i=1}^{n} \left(\frac{1}{2}\right)^{k-1} \\ &= n/2^{k-1} \\ &< 1. \end{aligned}$$

Explanation:
For any set $X \subseteq A$ and any $x \in \{R, B\}^X$ we have

$$\mathbf{P}(f(X) = x) = 2^{-|X|}.$$

1. The number of ω such that $f(X) = x$ is $2^{|A|-|X|}$.
2. $f(X) = x$ just depends on the random colours assigned to X and so is *independent* of colours not in X.

□

Example 3 A property of tournaments.

A tournament $T = (V, E)$ is an orientation of a complete graph. Suppose $V = [n]$. T has property A_k if for every $S \subseteq [n]$, $|S| = k$, there exists $w \notin S$ such that w "beats" S i.e. every edge vw with $v \in S$ is oriented from v to w. It seems quite difficult to construct tournaments with this property, especially if k is large.

Theorem 2. *If* $\binom{n}{k}\left(1 - \frac{1}{2^k}\right)^{n-k} < 1$ *then there exists a tournament with property* A_k.

Proof Let T be a random tournament on $[n]$ i.e. randomly orient the edges of K_n. For $S \subseteq [n]$, $|S| = k$, let

$$\mathcal{E}_S = \{\, \nexists v \notin S : \ v \text{ beats } S\}.$$

Then

$$\mathbf{Pr}(\mathcal{E}_S) = \left(1 - \frac{1}{2^k}\right)^{n-k}.$$

Here $1 - \frac{1}{2^k}$ is the probability that one $w \notin S$ fails to beat S and the $n-k$ events "v fails to beat S, $w \notin S$" are independent.

Thus

$$\begin{aligned}
\mathbf{Pr}(\neg A_k) &= \mathbf{Pr}\left(\bigcup_{\substack{S \subseteq [n] \\ |S|=k}} \mathcal{E}_S\right) \\
&\leq \sum_{\substack{S \subseteq [n] \\ |S|=k}} \mathbf{Pr}(\mathcal{E}_S) \\
&= \binom{n}{k}\left(1 - \frac{1}{2^k}\right)^{n-k} \\
&< 1.
\end{aligned}$$

□

Note that if we fix k and let $n \to \infty$, then $\binom{n}{k}\left(1 - \frac{1}{2^k}\right)^{n-k} \to 0$ and we say that a random tournament has property A_k, with high probability (**whp**) i.e. with probability tending to 1 as n tends to ∞.