Class 22

To see that the definition of PI makes sense we need to prove

Lemma 1. *If x, y are in the same orbit of X then $W(x) = W(y)$.*

Proof Suppose that $g * x = y$. Then

$$\begin{aligned} W(y) &= \prod_{d \in D} w_{y(d)} \\ &= \prod_{d \in D} w_{g*x(d)} \\ &= \prod_{d \in D} w_{x(g^{-1}(d))} \qquad (1) \\ &= \prod_{d \in D} w_{x(d))} \qquad (2) \\ &= W(x) \end{aligned}$$

Note, that we can go from (1) to (2) because as d runs over D, $g^{-1}(d)$ also runs over d. □

Let $\Delta = |D|$. If $g \in G$ has k_i cycles of length i then we define

$$ct(g) = x_1^{k_1} x_2^{k_2} \cdots x_\Delta^{k_\Delta}.$$

The **Cycle Index Polynomial** of G, C_G is then defined to be

$$C_G(x_1, x_2, \ldots, x_\Delta) = \frac{1}{|G|} \sum_{g \in G} ct(g).$$

In Example 2 with $n = 2$ we have

g	e	a	b	c	p	q	r	s
$ct(g)$	x_1^4	x_4	x_2^2	x_4	x_2^2	x_2^2	$x_1^2x_2$	$x_1^2x_2$

and so

$$C_G(x_1, x_2, x_3, x_4) = \frac{1}{8}(x_1^4 + 3x_2^2 + 2x_1^2 x_2 + 2x_4).$$

In Example 2 with $n = 3$ we have

g	e	a	b	c	p	q	r	s
$ct(g)$	x_1^9	$x_1x_4^2$	$x_1x_2^4$	$x_1x_4^2$	$x_1^3x_2^3$	$x_1^3x_2^3$	$x_1^3x_2^3$	$x_1^3x_2^3$

and so

$$C_G(x_1, x_2, x_3, x_4) = \frac{1}{8}(x_1^9 + x_1 x_2^4 + 4x_1^3 x_2^3 + 2x_1 x_4^2).$$

Theorem 1. *(***Polya***)*

$$PI = C_G\left(\sum_{c \in C} w_c, \sum_{c \in C} w_c^2, \ldots, \sum_{c \in C} w_c^\Delta\right).$$

In Example 2, we replace x_1 by $R+B$, x_2 by R^2+B^2 and so on. When $n=2$ this gives

$$\begin{aligned} PI &= \frac{1}{8}((R+B)^4 + 3(R^2+B^2) + 2(R+B)^2(R^2+B^2) + 2(R^4+B^4)) \\ &= R^4 + R^3B + 2R^2B^2 + RB^3 + B^4. \end{aligned}$$

Putting $R=B=1$ gives the number of distinct colourings. Note also the formula for PI tells us that there are 2 distinct colourings using 2 reds and 2 Blues.

Proof of Polya's Theorem
Let $X = X_1 \cup X_2 \cup \cdots \cup X_m$ be the equivalence clases of X under the relation

$$x \sim y \text{ iff } W(x) = W(y).$$

By Lemma 1, $g*x \sim x$ for all $x \in X, g \in G$ and so we can think of G acting on each X_i individually i.e. we use the fact that $x \in X_i$ implies $g*x \in X_i$ for all $i \in [m], g \in G$. We use the notation $g^{(i)} \in G^{(i)}$ when we restrict attention to X_i. Let m_i denote the number of orbits $\nu_{X_i,G^{(i)}}$. Then

$$\begin{aligned} PI &= \sum_{i=1}^{m} m_i W_i \\ &= \sum_{i=1}^{m} W_i \left(\frac{1}{|G|} \sum_{g\in G} |Fix(g^{(i)})| \right) \qquad \text{by Theorem the theorem of Burnside, Frobenius} \\ &= \frac{1}{|G|} \sum_{g\in G} \sum_{i=1}^{m} |Fix(g^{(i)})| W_i \\ &= \frac{1}{|G|} \sum_{g\in G} W(Fix(g)) \qquad (3) \end{aligned}$$

Note that (3) follows from $Fix(g) = \bigcup_{i=1}^{m} Fix(g^{(i)})$ since $x \in Fix(g^{(i)})$ iff $x \in W_i$ and $g*x = x$.

Suppose now that $ct(g) = x_1^{k_1} x_2^{k_2} \cdots x_\Delta^{k_\Delta}$ as above. Then we claim that

$$W(Fix(g)) = \left(\sum_{c\in C} w_c\right)^{k_1} \left(\sum_{c\in C} w_c^2\right)^{k_2} \cdots \left(\sum_{c\in C} w_c^\Delta\right)^{k_\Delta}. \qquad (4)$$

Substituting (4) into (3) yileds the theorem.

To verify (4) we use the fact that if $x \in Fix(g)$, then the elements of a cycle of g must be given the same colour. A cycle of length i will then contribute a factor $\sum_{c\in C} w_c^i$ where the term w_c^i comes from the choice of colour c for every element of the cycle. $\square$