Class 21

Cycles of a permutation

Let $\pi : D \to D$ be a permutation of the finite set D. Consider the digraph $\Gamma_\pi = (D, A)$ where $A = \{(i, \pi(i)) : i \in D\}$. Γ_π is a collection of vertex disjoint cycles. Each $x \in D$ being on a unique cycle. Here a cycle can consist of a loop i.e. when $\pi(x) = x$.

Example: $D = [10]$.

i	1	2	3	4	5	6	7	8	9	10
$\pi(i)$	6	2	7	10	3	8	9	1	5	4

The cycles are $(1, 6, 8), (2), (3, 7, 9, 5), (4, 10)$.

In general consider the sequence $i, \pi(i), \pi^2(i), \ldots,$. Since D is finite, there exists a first pair $k < \ell$ such that $\pi^k(i) = \pi^\ell(i)$. Now we must have $k = 0$, since otherwise putting $x = \pi^{k-1}(i) \neq y = \pi^{\ell-1}(i)$ we see that $\pi(x) = \pi(y)$, contradicting the fact that π is a permutation. So i lies on the cycle $C = (i, \pi(i), \pi^2(i), \ldots, \pi^{k-1}(i), i)$. If j is not a vertex of C then $\pi(j)$ is not on C and so we can repeat the argument to show that the rest of D is partitioned into cycles.

Example 1 First consider $e_0, e_1, \ldots, e_{n-1}$ as permutations of D. The cycles of e_0 are $(1), (2), \ldots, (n)$. Now suppose that $0 < m < n$. Let $a_m = gcd(m, n)$ and $k_m = n/a_m$. The cycle C_i of e_m containing the element i is is $(i, i+m, i+2m, \ldots, i+(k_m-1)m)$ since n is a divisor $k_m m$ and not a divisor of $k'm$ for $k' < k_m$. In total, the cycles of e_m are $C_0, C_1, \ldots, C_{a_m-1}$. This is because they are disjoint and together contain n elements. (If $i + rm = i' + r'm \mod n$ then $(r - r')m + (i - i') = \ell n$. But $|i - i'| < a_m$ and so dividing by a_m we see that we must have $i = i'$.)

Next observe that if colouring x is fixed by e_m then elements on the same cycle C_i must be coloured the same. Suppose for example that the colour of $i + bm$ is different from the colour of $i + (b+1)m$, say Red versus Blue. Then in $e_m(x)$ the colour of $i + (b+1)m$ will be Red and so $e_m(x) \neq x$. Conversely, if elements on the same cycle of e_m have the same colour then in $x \in Fix(e_m)$. This property is not peculiar to this example, as we will see.

Thus in this example we see that $|Fix(e_m)| = 2^{a_m}$ and then applying the Burnside/Frobenius Theorem we see that

$$\nu_{X,G} = \frac{1}{n} \sum_{m=0}^{n-1} 2^{gcd(m,n)}.$$

1 The pattern inventory

We now extend the above analysis to answer questions like: How many *distinct* ways are there to colour an 8×8 chessboard with 32 white squares and 32 black squares?

The scenrio now consists of a set D (*Domain*, a set C (colours) and $X = \{x : D \to C\}$ is the set of colourings of D with the colour set C. G is now a group of permutations of D.

We see first how to extend each permutation of D to a permutation of X. Suppose that $x \in X$ and $g \in G$ then we define $g * x$ by

$$g * x(d) = x(g^{-1}(d)) \qquad \text{for all } d \in D.$$

Explanation: The colour of d is the colour of the element $g^{-1}(d)$ which is mapped to it by g.

Consider Example 1 with $n = 4$. Suppose that $g = e_1$ i.e. rotate clockwise by $\pi/2$ and $x(1) = b, x(2) = b, x(3) = r, x(4) = r$. Then for example

$$g * x(1) = x(g^{-1}(1)) = x(4) = r, \text{ as before.}$$

Now associate a **weight** w_c with each $c \in C$. If $x \in X$ then

$$W(x) = \prod_{d \in D} w_{x(d)}.$$

Thus, if in Example 1 we let $w(r) = R$ and $w(b) = B$ and take $x(1) = b, x(2) = b, x(3) = r, x(4) = r$ then we will write $W(x) = B^2R^2$.

For $S \subseteq X$ we define the **inventory** of S to be

$$W(S) = \sum_{x \in S} W(x).$$

The probelm we discuss now is to compute the **pattern inventory** $PI = W(S^*)$ where S^* contains one member of each orbit of X under G.

For example, in the case of Example 2, with $n = 2$, we gt

$$PI = R^4 + R^3B + 2R^2B^2 + RB^3 + B^4.$$