Class 20

Let $\nu_{X,G}$ denote the number of orbits.

Theorem 1.

$$\nu_{X,G} = \frac{1}{|G|}\sum_{x\in X}|S_x|.$$

Proof

$$\begin{aligned}\nu_{X,G} &= \sum_{x\in X}\frac{1}{|O_x|}\\ &= \sum_{x\in X}\frac{|S_x|}{|G|},\end{aligned}$$

since $|O_x|\,|S_x| = |G|$ for all $x \in X$. □

Thus in example 1 we have

$$\nu_{X,G} = \frac{1}{4}(4+1+1+1+1+1+1+1+1+1+2+2+1+1+1+1+4) = 6.$$

In example 2 we have

$$\nu_{X,G} = \frac{1}{8}(8+2+2+2+2+2+2+2+2+4+4+2+2+2+2+8) = 6.$$

Theorem 1 is hard to use if $|X|$ is large, even if $|G|$ is small. For $g \in G$ let $Fix(g) = \{x \in X : g * x = x\}$.

Theorem 2. *(Frobenius, Burnside)*

$$\nu_{X,G} = \frac{1}{|G|}\sum_{g\in G}|Fix(g)|.$$

Proof Let $A(x,g) = 1_{g*x=x}$. Then

$$\begin{aligned}\nu_{X,G} &= \frac{1}{|G|}\sum_{x\in X}|S_x|\\ &= \frac{1}{|G|}\sum_{x\in X}\sum_{g\in G}A(x,g)\\ &= \frac{1}{|G|}\sum_{g\in G}\sum_{x\in X}A(x,g)\\ &= \frac{1}{|G|}\sum_{g\in G}|Fix(g)|.\end{aligned}$$

□

Let us consider example 1 with $n = 6$. We compute

g	e_0	e_1	e_2	e_3	e_4	e_5
$\|Fix(g)\|$	64	2	4	8	4	2

Applying Theorem 2 we obtain

$$\nu_{X,G} = \frac{1}{6}(64 + 2 + 4 + 8 + 4 + 2) = 14.$$

Example 2 It is straightforward to check that when n is even, we have

g	e	a	b	c	p	q	r	s
$\lvert Fix(g)\rvert$	2^{n^2}	$2^{n^2/4}$	$2^{n^2/2}$	$2^{n^2/4}$	$2^{n^2/2}$	$2^{n^2/2}$	$2^{n(n+1)/2}$	$2^{n(n+1)/2}$

For example, if we divide the chessboard into 4 $n/2 \times n/2$ sub-squares, numbered 1,2,3,4 then a colouring is in $Fix(a)$ iff each of these 4 sub-squares have the same colouring.