Class 19

Orbits: If $x \in X$ then its orbit $O_x = \{y \in X : \exists g \in G \text{ such that } g * x = y\}$.

Lemma 1. *The orbits partition X.*

Proof $x = 1_X * x$ and so $x \in O_x$ and so $X = \bigcup_{x \in X} O_x$.

Suppose now that $O_x \cap O_y \neq \emptyset$ i.e. $\exists g_1, g_2$ such that $g_1 * x = g_2 * y$. But then for any $g \in G$ we have

$$g * x = (g \circ (g_1^{-1} \circ g_2)) * y \in O_y$$

and so $O_x \subseteq O_y$. Similarly $O_y \subseteq O_x$. Thus $O_x = O_y$ whenever $O_x \cap O_y \neq \emptyset$. □

The two problems we started with are of the following form: Given a set X and a group of permutations *acting* on X, compute the number of orbits i.e. distinct colourings.

A subset H of G is called a *sub-group* of G if it satisfies *axioms* **A1,A2,A3** (with G replaced by H).

The *stabilizer* S_x of the element x is $\{g : g * x = x\}$. It is a sub-group of G.

Lemma 2.
If $x \in X$ then $|O_x|\,|S_x| = |G|$.

Proof Fix $x \in X$ and define an equivalence relation $\sim$ on G by

$$g_1 \sim g_2 \text{ if } g_1 * x = g_2 * x.$$

Let the equivalence classes be $A_1, A_2, \ldots, A_m$. We first argue that

$$|A_i| = |S_x| \qquad i = 1, 2, \ldots, m. \tag{1}$$

Fix i and $g \in A_i$. Then

$$h \in A_i \leftrightarrow g * x = h * x. \leftrightarrow (g^{-1} \circ h) * x = x \leftrightarrow (g^{-1} \circ h) \in S_x. \leftrightarrow h \in g \circ S_x$$

where $g \circ S_x = \{g \circ \sigma : \sigma \in S_x\}$. Thus $|A_i| = |g \circ S_x|$. But $|g \circ S_x| = |S_x|$ since if $\sigma_1, \sigma_2 \in S_x$ and $g \circ \sigma_1 = g \circ \sigma_2$ then $g^{-1} \circ (g \circ \sigma_1) = (g^{-1} \circ g) \circ \sigma_1 = \sigma_1 = g^{-1} \circ (g \circ \sigma_2) = \sigma_2$. This proves (1).

Finally, $m = |O_x|$ since there is a distinct equivalence class for each distinct $g * x$. □

Examples:

In example 1 with $n = 4$ we have

x	O_x	S_x
rrrr	$\{rrrr\}$	G
brrr	{brrr,rbrr,rrbr,rrrb}	$\{e_0\}$
rbrr	{brrr,rbrr,rrbr,rrrb}	$\{e_0\}$
rrbr	{brrr,rbrr,rrbr,rrrb}	$\{e_0\}$
rrrb	{brrr,rbrr,rrbr,rrrb}	$\{e_0\}$
bbrr	{bbrr,rbbr,rrbb,brrb}	$\{e_0\}$
rbbr	{bbrr,rbbr,rrbb,brrb}	$\{e_0\}$
rrbb	{bbrr,rbbr,rrbb,brrb}	$\{e_0\}$
brrb	{bbrr,rbbr,rrbb,brrb}	$\{e_0\}$
rbrb	{rbrb,brbr}	$\{e_0, e_2\}$
brbr	{rbrb,brbr}	$\{e_0, e_2\}$
bbbr	{bbbr,rbbb,brbb,bbrb}	$\{e_0\}$
bbrb	{bbbr,rbbb,brbb,bbrb}	$\{e_0\}$
brbb	{bbbr,rbbb,brbb,bbrb}	$\{e_0\}$
rbbb	{bbbr,rbbb,brbb,bbrb}	$\{e_0\}$
bbbb	{bbbb}	G

In example 2 we have

x	O_x	S_x
rrrr	$\{e\}$	G
brrr	{brrr,rbrr,rrbr,rrrb}	{e,r}
rbrr	{brrr,rbrr,rrbr,rrrb}	{e,s}
rrbr	{brrr,rbrr,rrbr,rrrb}	{e,r}
rrrb	{brrr,rbrr,rrbr,rrrb}	{e,s}
bbrr	{bbrr,rbbr,rrbb,brrb}	{e,p}
rbbr	{bbrr,rbbr,rrbb,brrb}	{e,q}
rrbb	{bbrr,rbbr,rrbb,brrb}	{e,p}
brrb	{bbrr,rbbr,rrbb,brrb}	{e,q}
rbrb	{rbrb,brbr}	{e,b,r,s}
brbr	{rbrb,brbr}	{e,b,r,s}
bbbr	{bbbr,rbbb,brbb,bbrb}	{e,s}
bbrb	{bbbr,rbbb,brbb,bbrb}	{e,r}
brbb	{bbbr,rbbb,brbb,bbrb}	{e,s}
rbbb	{bbbr,rbbb,brbb,bbrb}	{e,r}
bbbb	{e}	G