Class 18

Chapter 37: Polya's Theory of Counting

Example 1 A disc lies in a plane. Its centre is fixed but it is free to rotate. It has been divided into n sectors of angle $2\pi/n$. Each sector is to be coloured Red or Blue. How many different colourings are there? One could argue for 2^n. On the other hand, what if we only distinguish colourings which cannot be obtained from one another by a rotation. For example if $n = 4$ and the sectors are numbered 1,2,3,4 in clockwise order around the disc, then there are only 6 ways of colouring the disc – 4R, 4B, 3R1B, 1R3B, RRBB and RBRB.

Example 2 Now consider an $n \times n$ "chessboard" where $n \geq 2$. Here we colour the squares Red and Blue and two colourings are different only if one cannot be obtained from another by a rotation or a reflection. For $n = 2$ there are 6 colourings.

The general scenario that we consider is as follows: We have a set X which will stand for the set of colourings when transformations are not allowed. (In example 1, $|X| = 2^n$ and in example 2, $|X| = 2^{n^2}$). In addition there is a set G of permutations of X. You can think of the permutations as moving the colours around. The set of allowable permutations will have a group structure: Given two members $g_1, g_2 \in G$ we can define their composition $g_1 \circ g_2$ by $g_1 \circ g_2(x) = g_1(g_2(x))$ for $x \in X$. We require that G is *closed* under composiiton i.e. $g_1 \circ g_2 \in G$ if $g_1, g_2 \in G$. We also have the following:

A1 The *identity* permutation $1_X \in G$.

A2 $(g_1 \circ g_2) \circ g_3 = g_1 \circ (g_2 \circ g_3)$ (Composition is associative).

A3 The inverse permutation $g^{-1} \in G$ for every $g \in G$.

(A set G with a binary relation $\circ$ which satisfies **A1,A2,A3** is called a **Group**).

In example 1 $X = \{1, 2, \ldots, n\}$ the group is $G_1 = \{e_0, e_1, \ldots, e_{n-1}\}$ where $e_j * x = x + j \ mod \ n$ stands for rotation by $2j\pi/n$.

In example 2, $X = 2^{[n]^2}$. We number the squares 1,2,3,4 in clockwise order starting at the upper left and represent X as a sequence from $\{r, b\}^4$ where for example rrbr means colour 1,2,4 Red and 3 Blue. $G_2 = \{e, a, b, c, p, q, r, s\}$ is in a sense independent of n. e, a, b, c represent a rotation through $0, 90, 180, 270$ degrees respectively. p, q represent reflections in the vertical and horizontal and r, s represent reflections in the diagonals 1,3 and 2,4 respectively. Now check the following table:

	rrrr	brrr	rbrr	rrbr	rrrb	bbrr	rbbr	rrbb	brrb	rbrb	brbr	bbbr	bbrb	brbb	rbbb	bbbb
e	rrrr	brrr	rbrr	rrbr	rrrb	bbrr	rbbr	rrbb	brrb	rbrb	brbr	bbbr	bbrb	brbb	rbbb	bbbb
a	rrrr	rbrr	rrbr	rrrb	brrr	rbbr	rrbb	brrb	bbrr	brbr	rbrb	rbbb	bbbr	bbrb	brbb	bbbb
b	rrrr	rrbr	rrrb	brrr	rbrr	rrbb	brrb	bbrr	rbbr	rbrb	brbr	brbb	rbbb	bbbr	bbrb	bbbb
c	rrrr	rrrb	brrr	rbrr	rrbr	brrb	bbrr	rbbr	rrbb	brbr	rbrb	bbrb	brbb	rbbb	bbbr	bbbb
p	rrrr	rbrr	brrr	rrrb	rrbr	bbrr	brrb	rrbb	rbbr	brbr	rbrb	bbrb	bbbr	brbb	brbb	bbbb
q	rrrr	rrrb	rrbr	rbrr	brrr	rrbb	rbbr	bbrr	brrb	brbr	rbrb	rbbb	brbb	bbrb	bbbr	bbbb
r	rrrr	brrr	rrrb	rrbr	rbrr	brrb	rrbb	rbbr	bbrr	rbrb	brbr	brbb	bbrb	bbbr	rbbb	bbbb
s	rrrr	rrbr	rbrr	brrr	rrrb	rbbr	bbrr	brrb	rrbb	rbrb	brbr	bbbr	rbbb	brbb	bbrb	bbbb

From now on we will write $g * x$ in place of $g(x)$.

Orbits: If $x \in X$ then its orbit $O_x = \{y \in X : \exists g \in G \text{ such that } g * x = y\}$.

Lemma 1. *The orbits partition X.*

Proof $x = 1_X * x$ and so $x \in O_x$ and so $X = \bigcup_{x \in X} O_x$.

Suppose now that $O_x \cap O_y \neq \emptyset$ i.e. $\exists g_1, g_2$ such that $g_1 * x = g_2 * y$. But then for any $g \in G$ we have

$$g * x = (g \circ (g_1^{-1} \circ g_2)) * y \in O_y$$

and so $O_x \subseteq O_y$. Similarly $O_y \subseteq O_x$. Thus $O_x = O_y$ whenever $O_x \cap O_y \neq \emptyset$. □

The two problems we started with are of the following form: Given a set X and a group of permutations *acting* on X, compute the number of orbits i.e. distinct colourings.