Class 17

Principle of Inclusion-Exclusion

Euler's Function $\phi(n)$.

Let $\phi(n)$ be the number of positive integers $x \leq n$ which are mutually prime to n i.e. have no common factors with n, other than 1.

$\phi(12) = 4$.

Let $n = p_1^{\alpha_1} p_2^{\alpha_2} p_1^{\alpha_2} \cdots p_k^{\alpha_k}$ be the prime factorisation of n.

$$A_i = \{x \in [n] : \ p_i \text{ divides } x\}, \qquad 1 \leq i \leq k.$$

$$\phi(n) = \left| \bigcap_{i=1}^{k} \bar{A}_i \right|$$

$$|A_S| = \frac{n}{\prod_{i \in S} p_i} \qquad S \subseteq [k].$$

$$\begin{aligned} \phi(n) &= \sum_{S \subseteq [k]} (-1)^{|S|} \frac{n}{\prod_{i \in S} p_i} \\ &= n \left(1 - \frac{1}{p_1}\right) \left(1 - \frac{1}{p_2}\right) \cdots \left(1 - \frac{1}{p_k}\right) \end{aligned}$$

Silly Example

Let $X = \{f : [n] \to [n] : \ f(i) \neq i \text{ for } i = 1, 2, \ldots, n\}$. Now $|X| = (n-1)^n$ since there are $n-1$ choices for $f(i)$, for each i.

We can also practise inclusion-exclusion with this example. Let $A_i = \{f : [n] \to [n] : \ f(i) = i\}$ $i = 1, 2, \ldots, n$. Then $X = \bigcap_{i=1}^n \bar{A}_i$. So,

$$\begin{aligned} |X| &= \sum_{S \subseteq [n]} (-1)^{|S|} |A_S| \\ &= \sum_{S \subseteq [n]} (-1)^{|S|} n^{n-|S|} \\ &= \sum_{k=0}^{n} \binom{n}{k} (-1)^k n^{n-k} \\ &= n^n \sum_{k=0}^{n} \binom{n}{k} (-1)^k n^{-k} \end{aligned}$$

(by the binomial theorem)

$$\begin{aligned} &= n^n \left(1 - \frac{1}{n}\right)^n \\ &= (n-1)^n. \end{aligned}$$

Proofs of inclusion-exclusion formula

Proof 1

$$\theta_{x,i} = \begin{cases} 1 & x \in A_i \\ 0 & x \notin A_i \end{cases}$$

Then

$$(1-\theta_{x,1})(1-\theta_{x,2})\cdots(1-\theta_{x,N}) = \begin{cases} 1 & x \in \bigcap_{i=1}^{N} \bar{A}_i \\ 0 & \text{otherwise} \end{cases}$$

So

$$\begin{aligned}
\left|\bigcap_{i=1}^{N} \bar{A}_i\right| &= \sum_{x\in A}(1-\theta_{x,1})(1-\theta_{x,2})\cdots(1-\theta_{x,N}) \\
&= \sum_{x\in A}\sum_{S\subseteq[N]}(-1)^{|S|}\prod_{i\in S}\theta_{x,i} \\
&= \sum_{S\subseteq[N]}(-1)^{|S|}\sum_{x\in A}\prod_{i\in S}\theta_{x,i} \\
&= \sum_{S\subseteq[N]}(-1)^{|S|}|A_S|.
\end{aligned}$$

Proof 2 Write the formula as

$$\left|\bigcap_{i=1}^{N} \bar{A}_i\right| = n - n_1 + n_2 - n_3 + \cdots + (-1)^N n_N \tag{1}$$

where

$$n_i = \sum_{\substack{S\subseteq[N] \\ |S|=i}} |A_S| \qquad \text{for } i = 1, 2, \ldots, N.$$

If x in in none of the A_i then x contributes 1 to the RHS of (1).

If x is in exactly $k \geq 1$ of the A_i then x contributes

$$1 - \binom{k}{1} + \binom{k}{2} - \binom{k}{3} + \cdots + (-1)^k\binom{k}{k} = (1-1)^k = 0$$

to the RHS of (1). □