Class 15

Dilworth's theorem and extremal set theory

We consider the size of the largest anti-chain $\mathcal{A}$ in the power set $\mathcal{P}_n = \{A : A \subseteq [n]\}$.

Theorem 1. Sperner
If $\mathcal{A} \subseteq \mathcal{P}_n$ is an ant–chain then $|\mathcal{A}| \leq \binom{n}{\lfloor n/2 \rfloor}$.

Proof Let $\alpha_k = |\{A \in \mathcal{A} : |A| = k\}|$ be the number of k-sets in $\mathcal{A}$. We show that

$$\sum_{k=0}^{n} \frac{\alpha_k}{\binom{n}{k}} \leq 1. \tag{1}$$

And then because $\binom{n}{k} \leq \binom{n}{\lfloor n/2 \rfloor}$ for all k we have

$$\sum_{k=0}^{n} \frac{\alpha_k}{\binom{n}{\lfloor n/2 \rfloor}} \text{ and so } |\mathcal{A}| = \sum_{k=0}^{n} \alpha_k \leq \binom{n}{\lfloor n/2 \rfloor}.$$

To verify (1) we define a maximal chain C_π for every permutation π of $[n]$. Here

$$C_\pi = (\emptyset \subseteq \{\pi(1)\} \subseteq \{\pi(1), \pi(2)\} \subseteq \cdots \subseteq [n].$$

Then for permutation π and $A \in \mathcal{A}$ define

$$M(\pi, A) = \begin{cases} 1 & A \in C_\pi \\ 0 & \text{otherwise} \end{cases}$$

Then we have

$$\sum_{A \in \mathcal{A}} M(\pi, A) \leq 1 \qquad \text{for all permutations } \pi \tag{2}$$

$$\sum_{\pi} M(\pi, A) = |A|!(n - |A|)! \qquad \text{for all } A \in \mathcal{A} \tag{3}$$

(2) follows from the fact that $\mathcal{A}$ is an anti-chain and any 2 sets in C_π are comparable.
(3) is explained as follows: Suppose that $|A| = k$. Then there are $k!$ ways of ordering the elements of A so that it is the k-element set in C_π and then for each such ordering, there are $(n-k)!$ ways of ordering the rest of the elements of $[n]$.

Then we have

$$n! \geq \sum_{\pi} \sum_{A \in \mathcal{A}} M(\pi, A) = \sum_{A \in \mathcal{A}} \sum_{\pi} M(\pi, A) = \sum_{A \in \mathcal{A}} |A|!(n - |A|)! = \sum_{k=0}^{n} \alpha_k k!(n-k)!$$

and (1) follows on dividing $\sum_{k=0}^{n} \alpha_k k!(n-k)! \leq n!$ by $n!$. □

Intersecting Families A family $\mathcal{A} \subseteq \mathcal{P}_n$ is an *intersecting* family if $A, B \in \mathcal{A}$ implies $A \cap B \neq \emptyset$.

Exercise If $\mathcal{A}$ is an intersecting family then $|\mathcal{A}| \leq 2^{n-1}$.

Theorem 2. Bollobás, Erdős-Ko-Rado
If $\mathcal{A}$ is an intersecting family which is also an anti-chain and $A \in \mathcal{A}$ implies that $|A| \leq \lfloor n/2 \rfloor$ then

$$\sum_{k=1}^{\lfloor n/2 \rfloor} \frac{\alpha_k}{\binom{n-1}{k-1}} \leq 1. \tag{4}$$

Proof If π is a permutation of $[n]$ then an *interval* of π is a set of the form $\{\pi(i), \pi(i+1), \ldots, \pi(i+j)\}$ where $\pi(i+k) = \pi(i+k-n)$ if $i+k > n$. For $A \in \mathcal{A}$ we will write $A \in \pi$ to mean that A is an interval of π.

Now define

$$M(\pi, A) = \begin{cases} \frac{1}{|A|} & A \in \pi \\ 0 & \text{otherwise} \end{cases}$$

Then we have

$$\sum_{A\in\mathcal{A}} M(\pi, A) \quad \leq \quad 1 \qquad \text{for all permutations } \pi \tag{5}$$

$$\sum_{\pi} M(\pi, A) \quad = \quad n(|A|-1)!(n-|A|)! \qquad \text{for all } A \in \mathcal{A} \tag{6}$$

(5) is explained as follows: Let k be the size of the smallest interval of π which is a member of $\mathcal{A}$. Assume w.l.o.g. that $A = [k]$ and $\pi(i) = i$ for $i \in [n]$. If $A' \in \mathcal{A}$ is another interval in π, let it be of type 1 if it contains 1 and of type 2 otherwise. A type 1 interval is of the form $\{x, x+1, \ldots, n, 1, \ldots, i\}$, $i \neq k$ and a type 2 interval is of the form $\{i, i+1, \ldots, k, k+1, \ldots, y\}$, $i \neq 1$. Here i is calld the *edge* of the interval. There is at most one type 1 interval with edge i, else $\mathcal{A}$ is not an anti-chain. Similarly for type 2 intervals. Finally, if there is a type 1 and a type 2 interval with the same edge i then there are at most $k-i$ type 1 intervals and at most $i-1$ type 2 intrvals. Finally note that each of the at most k intervals in π contributes at most $1/k$.

(6) is explained as follows: Suppose that $|A| = k$. Then there n choices for the start of the interval and then there are $k!$ ways of ordering the elements of A so that it is interval following then for each such ordering, there are $(n-k)!$ ways of ordering the rest of the elements of $[n]$. Then we divide by $|A| = k$.

Then we have

$$n! \geq \sum_{\pi} \sum_{A\in\mathcal{A}} M(\pi, A) = \sum_{A\in\mathcal{A}} \sum_{\pi} M(\pi, A) = \sum_{A\in\mathcal{A}} n(|A|-1)!(n-|A|)! = \sum_{k=1}^{\lfloor n/2 \rfloor} \alpha_k n(k-1)!(n-k)!$$

and (4) follows on dividing $\sum_{k=1}^{\lfloor n/2 \rfloor} \alpha_k n(k-1)!(n-k)! \leq n!$ by $n!$ □

Corollary 3. *If $\mathcal{A} \subseteq \mathcal{P}_n$ is intersecting and if $A \in \mathcal{A}$ implies that $|A| = k \leq \lfloor n/2 \rfloor$ then $|\mathcal{A}| \leq \binom{n-1}{k-1}$.*