**Class 14**

Dilworth's theorem and extremal set theory

**Theorem 1.** *(Dilworth) Let $P$ be a finite poset, then*
$\min\{m :\ \exists$ *chains* $C_1, C_2, \ldots, C_m$ *with* $P = \bigcup_{i=1}^m C_i\} = \max\{|A| :\ A$ *is an anti-chain*$\}$.

**Proof** We have already argued that $\max\{|A|\} \geq \min\{m\}$. We will prove there is equality here by induction on $|P|$. The result is trivial if $|P| = 0$.

Now assume that $|P| > 0$ and the $m$ is the maximum size of an anti-chain in $P$. Let $C = x_1 < x_2 < \cdots < x_p$ be a *maximal* chain in $P$ i.e. we cannot add elements to it and keep it a chain.

**Case 1** Every anti-chain in $P \setminus C$ has $\leq m - 1$ elements. Then by induction $(P \setminus C) = \bigcup_{i=1}^{m-1} C_i$ and then $P = C \cup \bigcup_{i=1}^{m-1} C_i$ and we are done.

**Case 2** There exists an anti-chain $A = \{a_1, a_2, \ldots, a_m\}$ in $P \setminus C$. Let

- $P^- = \{x \in P :\ x \leq a_i \text{ for some } i\}$.
- $P^+ = \{x \in P :\ x \geq a_i \text{ for some } i\}$.

Note that

1. $P = P^- \cup P^+$. Otherwise there is an element $x$ of $P$ which is incomparable with every element of $A$ and so $m$ is not the maximum size of an anti-chain, contradiction.
2. $P^- \cap P^+ = A$. Otherwise there exists $x, i, j$ such that $a_i < x < a_j$ and so $A$ is not an anti-chain.
3. $x_p \notin P^-$. Otherwise $x < a_i$ for some $i$ and the chain $C$ is not maximal.

Applying the inductive hypothesis to $P^-$ ($|P^-| < |P|$ follows from 3) we see that $P^-$ can be partitioned into $m$ chains $C_1^-, C_2^-, \ldots, C_m^-$. Now the elements of $A$ must be distributed one to a chain and so we can assume that $a_i \in C_i^-$ for $i = 1, 2, \ldots, m$. Furthermore, $a_i$ must be the maximum element of chain $C_i^-$, else the maximum of $C_i^-$ is in $P^- \cap P^+ \setminus A$, which contradicts 2.

Applying the same argument to $P^+$ we get chains $C_1^+, C_2^+, \ldots, C_m^+$ with $a_i$ as the minimum element of $C_i^+$ for $i = 1, 2, \ldots, m$. Then from 2. we see that $P = C_1 \cup C_2 \cup \cdots \cup C_m$ where $C_i = C_i^- \cup C_i^+$ is a chain for $i = 1, 2, \ldots, m$. $\square$

**Theorem 2.** *Let $P$ be a finite poset, then*
$\min\{m :\ \exists$ *anti-chains* $A_1, A_2, \ldots, A_m$ *with* $P = \bigcup_{i=1}^m A_i\} = \max\{|C| :\ A$ *is a chain*$\}$.

**Proof** The minimum number of anti-chains needed to cover $P$ is at least the size of any chain, since a chain can contain at most one element from each anti-chain.

We prove the converse by induction on the maximum length $m$ of a chain. If $m = 1$ then $P$ itself is an anti-chain and this provides the basis of the induction.

So now suppose that $C = x_1 < x_2 < \cdots < x_m$ be a maximum length chain and let $A$ be the set of maximal elements of $P$ – an element is $x$ m*maximal* if $\nexists y$ such that $y > x$. Now consider $P' = P \setminus A$. $P'$ contains no chain of length $m$. If it contained $y_1 < y_2 < \cdots < y_m$ then since $y_m \notin A$, there exists $a \in A$ such that $P$ contains the chain $y_1 < y_2 < \cdots < y_m < a$, contradiction. Thus the maximum length of a chain in $P'$ is $m - 1$ and so it can be partitioned into anti-chains $A_1 \cup A_2 \cup \cdots A_{m-1}$. Putting $A_m = A$ completes the proof. $\square$

**Two applications of Dilworth's Theorem**

(i) Let $a_1, a_2, \ldots, a_{n^2+1}$ be a sequence of real numbers. A sub-sequence $i_1 < i_2 < \cdots i_k$ is said to be *monotone increasing* (resp. *monotone decreasing* if $a_{i_1} \leq a_{i_2} \leq \cdots \leq a_{i_k}$ (resp. $a_{i_1} \geq a_{i_2} \geq \cdots \geq a_{i_k}$. It is monotone if it is monotone increasing or decreasing.

**Theorem 3.** *Erdős and Szekerés*
*$a_1, a_2, \ldots, a_{n^2+1}$ contains a monotone subsequence of length $n+1$.*

**Proof** Let $P = \{(i, a_i) : 1 \leq i \leq n^2 + 1\}$ and let say $(i, a_i) \leq (j, a_j)$ if $i < j$ and $a_i \leq a_j$. A chain in $P$ corresponds to a monotone increasing subsequence. So, suppose that there are no monotone increasing sequences of length $n+1$. Then any cover of $P$ by chains requires at least $n+1$ chains and so, by Dilworths theorem, there exists an anti-chain $A$ of size $n+1$.

Let $A = \{(i_t, a_{i_t}) : 1 \leq t \leq n+1\}$ where $i_1 < i_2 \leq \cdots < i_{n+1}$. Finally observe that $a_{i_t} > a_{i_{t+1}}$ for $1 \leq t \leq n$, for otherwise $(i_t, a_{i_t}) \leq (i_{t+1}, a_{i_{t+1}})$ and $A$ is not an anti-chain. Thus $A$ defines a monotone decreasing sequence of length $n+1$. □

(ii) Another proof of Hall's Theorem. Let $G = (X \cup Y, E)$ be a bipartite graph which satisfies Hall's condition. Define a poset $P = X \cup Y$ and define $<$ by $x < y$ only if $x \in X, y \in Y$ and $(x, y) \in E$.

Suppose that the largest anti-chain in $P$ is $A = \{x_1, x_2, \ldots, x_h, y_1, y_2, \ldots, y_k\}$ and let $s = h + k$. Now

$$\Gamma(\{x_1, x_2, \ldots, x_h\}) \subseteq Y \setminus \{y_1, y_2, \ldots, y_k\}$$

for otherwise $A$ will not be an anti-chain.

From Hall's condition we see that

$$|Y| - k \geq h \text{ or } |Y| \geq s.$$

Now by Dilworth's theorem, $P$ is the union of $s$ chains: A matching $M$ of size $m$, $|X| - m$ members of $X$ and $|Y| - m$ members of $Y$. But then

$$m + (|X| - m) + (|Y| - m) = s \leq |Y|$$

and so $m \geq |X|$. □