Class 13

Dilworth's theorem and extremal set theory

A **partially ordered set** or **poset** is a set P and a binary relation $\leq$ such that for all $a, b, c \in P$

1. $a \leq a$ (reflexivity).
2. $a \leq b$ and $b \leq c$ implies $a \leq c$ (transitivity).
3. $a \leq b$ and $b \leq a$ implies $a = b$. (anti-symmetry).

Examples

1. $P = \{1, 2, \ldots, \}$ and $a \leq b$ has the usual meaning.
2. $P = \{1, 2, \ldots, \}$ and $a \leq b$ if a divides b.
3. $P = \{A_1, A_2, \ldots, A_m\}$ where the A_i are sets and $\leq = \subseteq$.

A pair of elements a, b are **comparable** if $a \leq b$ or $b \leq a$. Otherwise they are **incomparable**.

A poset without incomparable elements (Example 1) is a linear or total order.

We write $a < b$ if $a \leq b$ and $a \neq b$.

A **chain** is a sequence $a_1 < a_2 < \cdots < a_s$.

A set A is an **anti-chain** if every pair of elements in A are incomparable.

Suppose that $C_1, C_2, \ldots, C_k$ are a collection of chains such that $P = \bigcup_{i=1}^{k} C_i$ (think of $a_1 < a_2 < \cdots < a_s$ as the set $\{a_1, a_2, \ldots, a_s\}$ here). Suppose that A is an anti-chain. Then $k \geq |A|$ because if $k < |A|$ then by the pigeon-hole principle there will be two elemtns of A in some chain.

Theorem 1. *(Dilworth) Let P be a finite poset, then*
$\min\{m : \exists$ *chains* $C_1, C_2, \ldots, C_m$ *with* $P = \bigcup_{i=1}^{m} C_i\} = \max\{|A| :$ *A is an anti-chain*$\}$.

Proof We have already argued that $\max\{|A|\} \geq \min\{m\}$. We will prove there is equality here by induction on $|P|$. The result is trivial if $|P| = 0$.

Now assume that $|P| > 0$ and the m is the maximum size of an anti-chain in P. Let $C - x_1 < x_2 < \cdots < x_p$ be a *maximal* chain in P i.e. we cannot add elements to it and keep it a chain.

Case 1 Every anti-chain in $P \setminus C$ has $\leq m - 1$ elements. Then by induction $(P \setminus C) = \bigcup_{i=1}^{m-1} C_i$ and then $P = C \cup \bigcup_{i=1}^{m-1} C_i$ and we are done.

Case 2 There exists an anti-chain $A = \{a_1, a_2, \ldots, a_m\}$ in $P \setminus C$. Let

- $P^- = \{x \in P : \; x \leq a_i$ for some $i\}$.
- $P^+ = \{x \in P : \; x \geq a_i$ for some $i\}$.

Note that $P = P^- \cup P^+$. Otherwise there is an element x of P which is incomparable with every element of A and so m is not the mximum size of an anti-chain, contradiction.

To be continued