Class 11

Chapter 5: Systems of distinct representatives

Consider a bipartite graph $G = (X \cup Y, E)$ where all edges have one vertex in X and one vertex in Y.

A *matching* M of G is a set of edges which are pair-wise disjoint i.e. no vertex is on two or more edges of M.

For $A \subseteq X$ we let $\Gamma(A) = \{y \in Y : \exists x \in X \text{ such that } (x, y) \in E\}$.

A *complete* matching M from X to Y is a matching which *covers* the whole of X. (M covers v if M contains an edges e for which $v \in e$).

Theorem 1. Hall's Theorem
G has a complete matching from X to Y iff

$$|\Gamma(A)| \geq |A| \qquad \forall A \subseteq X. \tag{1}$$

Proof **Only if** Suppose that $\{(x, f(x)) : x \in X\}$ is a complete matching of X into Y. Then f here is a 1-1 function of X into Y and for $A \subseteq X$ we have

$$|\Gamma(A)| \geq |f(A)| = |A|.$$

If Suppose that $|X| = n$ and $M, |M| = m < n$ is a matching of G. We show that there exists a matching with $m + 1$ edges.

We colour the edges of M Red and the remaining edges Blue. Since $|M| < |X|$ there exist vertices in X which are not covered by M. Let x_0 be one such vertex.

An *alternating path* is a path P, starting at X, whose edges are alternatively Blue,Red,Blue,Red,... .

Let $X' \subseteq X$, $Y' \subseteq Y$ be those vertices which are reachable from x_0 by alternating paths.

We claim that Y' has a vertex y^* which is not covered by M. Suppose not.

Then $y \in Y'$ implies there is a Red edge $(g(y), y)$ and if $y \neq y' \in Y'$ then $g(y) \neq g(y')$. Now if $y \in Y'$ then $g(y) \in X'$, since an alternating path to y must end in a Blue edge (only odd length paths send in Y) and if it does not contain the vertex $g(y)$ then we can add the Red edge $(g(y), y)$. Thus

$$|X'| \geq 1 + |g(Y')| = 1 + |Y'|$$

where the 1 comes from $x_0 \in X'$.

On the other hand we can see that

$$\Gamma(X') \subseteq Y'$$

and this contrdicts our assumption that (1) holds.

(If $x \in X'$ and P is an alternating path to x and y is a neighbour of x which is not in Y' then we can extend P by adding the Blue edge (x, y). (P must end in a Red edge – only even length paths send in X)).

Thus y^* exists and so let $P = (x_0, y_1, x_1, y_2, \ldots, x_k, y_{k+1} = y^*)$ be an alternating path from x_0 to y^*. The

$$M' = M + (x_0, y_1) \cup \{(x_1, y_2) + \cdots + (x_k, y_{k+1})\} - \{(y_1, x_1) + \cdots + (y_k, x_k)\}$$

is a matching of size $m + 1$ in G.

(In going from M to M' x_0 and y^* are covered exactly once and the remaining vertices of P have their matching edges replaced i.e. they are still incident with exactly one vertex. Vertices not on P are not affected).