Class 10

Parallel searching for the maximum – Valiant

We have n processors and n numbers $x_1, x_2, \ldots, x_n$. In each round we choose n pairs i, j and compare the values of x_i, x_j.
The set of pairs chosen in a round can depend on the results of previous comparisons.

Aim: find i^* such that $x_{i^*} = \max_i x_i$.

Claim 1. *For any algorithm there exists an input which requires at least* $\frac{1}{2}\log_2\log_2 n$ *rounds.*

Let $C_{\mathcal{A}}(a, b)$ be the maximum number of rounds needed by algorithm $\mathcal{A}$ which uses a processors to compute the maximum of b values in this way. The maximum is over the inputs defined by the adversary A.

Lemma 1.

$$C_{\mathcal{A}}(a, b) \geq 1 + C_{\mathcal{A}}\left(a, \left\lceil \frac{b^2}{2a+b} \right\rceil\right).$$

Proof The set of b comparisons chosen by $\mathcal{A}$ defines a b-edge graph G on a vertices where comparison of x_i, x_j produces an edge (i, j) of G. Theorem 1 (proved below) implies that the the maximum size $\alpha(G)$ of an *independent* set of G satisfies

$$\alpha(G) \geq \left\lceil \frac{b}{\frac{2a}{b}+1} \right\rceil = \left\lceil \frac{b^2}{2a+b} \right\rceil.$$

(A set of vertices is independent if it contains no edges)

For any independent set I it is always possible for the adversary A to define values for $x_1, x_2, \ldots, x_a$ such that I is the index set of the $|I|$ largest values and so that the comparisons do not yield any information about the ordering of the elements $x_i, i \in I$.

Thus after one round one has the problem of finding the maximum among at least $\alpha(G)$ elements. Any comparison involving a vertex outside the maximum independent set chosen by A is wasted. $\Box$

Now define the sequence $c_0, c_1, \ldots$ by $c_0 = n$ and

$$c_{i+1} = \left\lceil \frac{c_i^2}{2n + c_i} \right\rceil.$$

It follows from Lemma 1 that

$$c_k \geq 2 \text{ implies } C(n, n) \geq k + 1.$$

Claim 2.

$$c_i \geq \frac{n}{3^{2^i - 1}}.$$

By induction on i. Trivial for $i = 0$. Then

$$\begin{aligned} c_{i+1} &\geq \frac{n^2}{3^{2^{i+1}-2}} \times \frac{1}{2n + n3^{2^i - 1}} \\ &= \frac{n}{3^{2^{i+1}-1}} \times \frac{3}{2 + \frac{1}{3^{2^i - 1}}} \\ &\geq \frac{n}{3^{2^{i+1}-1}}. \end{aligned}$$

$\square$

Claim 1 now follows from the the above claim since

$$3^{2^{\frac{1}{2}\log_2\log_2 n}} = 3^{(\log_2 n)^{1/2}} = o(n).$$

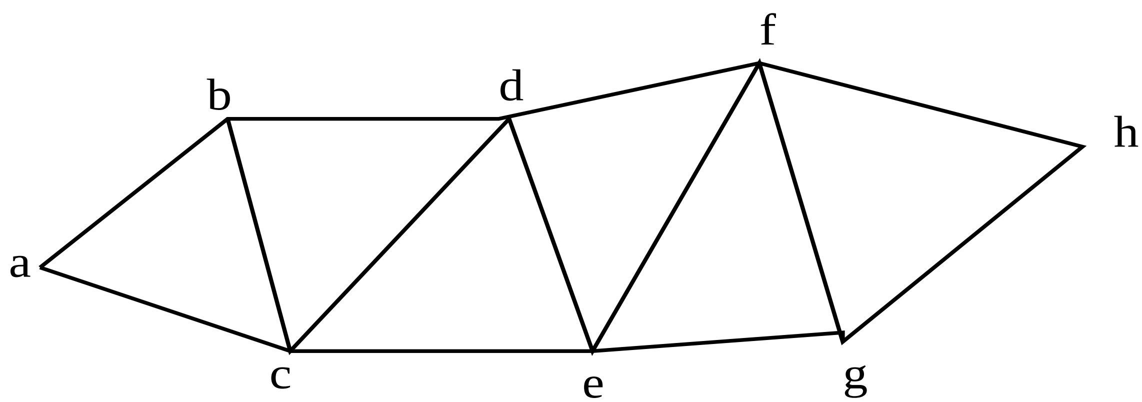
Theorem 1. *If $\bar{d} = 2m/n$ = the average degree of simple graph G with n vertices and m edges then*

$$\alpha(G) \geq \frac{n}{\bar{d}+1}.$$

Proof Let $\pi(1), \pi(2), \ldots, \pi(\nu)$ be an arbitrary permutation of V. Let $N(v)$ denote the set of neighbours of vertex v and let

$$I(\pi) = \{v : \pi(w) > \pi(v) \text{ for all } w \in N(v)\}.$$

Claim 3. *I is an independent set.*



	a	b	c	d	e	f	g	h	I
π_1	c	b	f	h	a	g	e	d	$\{c, f\}$
π_2	g	f	h	d	e	a	b	c	$\{g, d, a\}$

Proof of Claim 3
Suppose $w_1, w_2 \in I(\pi)$ and $w_1w_2 \in E$. Suppose $\pi(w_1) < \pi(w_2)$. Then $w_2 \notin I(\pi)$ — contradiction. $\square$

Now let π be a random permutation.

Claim 4.

$$\mathbf{E}(|I|) \geq \sum_{v \in V} \frac{1}{d(v)+1}.$$

Proof of Claim 4
Let

$$\delta(v) = \begin{cases} 1 & v \in I \\ 0 & v \notin I \end{cases}$$

Thus

$$\begin{aligned} |I| &= \sum_{v \in V} \delta(v) \\ \mathbf{E}(|I|) &= \sum_{v \in V} \mathbf{E}(\delta(v)) \\ &= \sum_{v \in V} \mathbf{Pr}(\delta(v) = 1). \end{aligned}$$

Now $\delta(v) = 1$ if v comes before all of its neighbours in the order π. Thus

$$\mathbf{Pr}(\delta(v) = 1) = \frac{1}{d(v)+1}$$

and the claim follows. $\Box$

(Note that as defined, I is not maximal i.e. it may be possible to add more vertices to I while preserving independence. Also, here I is contained in the independent set J chosen by a *Greedy* algorithm: Go through the vertice in the order specified by the permutation. Put v into J if none of its neighbours have been already been placed in J.)

Thus there exists a π such that

$$|I(\pi)| \geq \sum_{v \in V} \frac{1}{d(v)+1}$$

and so

$$\alpha(G) \geq \sum_{v \in V} \frac{1}{d(v)+1}.$$

We finish the proof of Theorem 1 by showing that

$$\sum_{v \in V} \frac{1}{d(v)+1} \geq \frac{n}{\bar{d}+1}.$$

This follows from the following claim by putting $x_v = d(v) + 1$ for $v \in V$.

Claim 5. *If* $x_1, x_2, \ldots x_k > 0$ *then*

$$\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_k} \geq \frac{k^2}{x_1 + x_2 + \cdots + x_k}. \tag{1}$$

Proof of Claim 5

Multiplying (1) by $x_1 + x_2 + \cdots + x_k$ and subtracting k from both sides we see that (1) is equivalent to

$$\sum_{1 \leq i < j \leq k} \left(\frac{x_i}{x_j} + \frac{x_j}{x_i} \right) \geq k(k-1). \tag{2}$$

But for all $x, y > 0$

$$\frac{x}{y} + \frac{y}{x} \geq 2$$

and (2) follows. $\Box$