Class 09

Theorem 1. *Let $N = N(m, m; 3)$. Let $X_1, X_2, \ldots, X_N$ be points in the plane such that no three are collinear. Then there exist m points which form a convex polygon.*

Proof For $i, j, k \in [N]$ let $|ijk|$ denote the number of points contained entirely within the triangle $X_i X_j X_k$. Now colour $\{i, j, k\}$ with 0 if $|ijk|$ is even and 1 if $|ijk|$ is odd.

By Ramsey's theorem there exists a subset A of $[N]$ such that all triples in A have the same colour. The corresponding points form a convex polygon. If not, there exist four points X_i, X_j, X_k, X_l such that X_l lies entirely inside the triangle $X_i X_j X_k$. But then

$$|ijk| = |ikl| + |ijl| + |jkl| + 1$$

which is impossible if the four values $|ijk|, |ikl|, |ijl|, |jkl|$ have the same parity.

Turán's Theorem and extremal graphs.

We consider the number of edges that a graph G on n vertices can have without containing of K_p, $p \geq 1$ as a subgraph.

Suppose that

$$n = t(p-1) + r \qquad \text{where } 1 \leq r \leq p-1.$$

Define the Turán graph $T(n, p)$ with vertex set $[n]$ as follows: Partition $[n]$ into sets $S_1, S_2, \ldots, S_{p-1}$ where

$$|S_i| = \begin{cases} t+1 & 1 \leq i \leq r \\ t & r+1 \leq i \leq p-1 \end{cases}.$$

Create an edge for every pair of vertices in distinct sets. The number of edges in $T(n, p)$ is then $M(n, p)$ where

$$M(n,p) = \binom{r}{2}(t+1)^2 + \binom{p-1-r}{2}t^2 + r(p-1-r)t(t+1).$$

Now $T(n, p)$ cannot contain a copy of K_p. For any p vertices must contain two vertices from the same S_i and these two vertices are no adjacent in $T(n, p)$.

Theorem 2. *(Turán) If a simple graph on n vertices has more than $M(n, p)$ edges then it must contain K_p as subgraph.*

Proof We fix $p \geq 1$ and prove the theorem by induction on t.

If $t = 0$ then $n = r \leq p - 1$ and $M(n, p) = \binom{r}{2}$. In this case the statement of the theorem is (vacuously) true since a graph on r vertices cannot have more than $M(n, p)$ edges.

Now consider a graph G with n vertices and no copy of K_p and **with as many edges as possible** under these conditions. Then G contains a copy H of K_{p-1}, else we can add an edge to G without creating a copy of K_p.

Now each $v \notin V(H)$ has at most $p-2$ neighbours in H, else G has a copy of K_p. Also, the subgraph induced by $V(G) \setminus V(H)$ has $n-p+1$ vertices and no copy of K_p. Since $n-p+1 = (t-1)(p-1)+r$ our inductive hypothesis implies that this subgraph has at most $M(n-p+1, p)$ edges, by induction. It follows that the number of edges in G is at most

$$M(n-p+1, p) + (p-2)(n-p+1) + \binom{p-1}{2} = M(n, p).$$

This can be checked by tedious, but elementary methods.

Back to the base case: If it seems that we didn't prove anything, think of how we use the base case to verify the case $t = 1$. Here $n - p + 1 = r \leq p - 1$ and all we require to proceed is that the number of edges inside $V \setminus V(H)$ is at most $M(n, p) = \binom{r}{2}$, which is obvious anyway.