**Class 08**

Colorings of graphs and Ramsey's Theorem

## Ramsey's Theorem in general

**Theorem 1.** *Let $r, s \geq 1$, $q_i \geq r$, $1 \leq i \leq s$ be given. Then there exists $N = N(q_1, q_2, \ldots, q_s; r)$ with the following property: Suppose that $S$ is a set with $n \geq N$ elements. Let each of the elements of $\binom{S}{r}$ be given one of $s$ colours. (***Note that the elements of $\binom{S}{r}$ are the $r$-subsets of** *$S$).*

*Then there exists $i$ and a $q_i$-subset $T$ of $S$ such that all of the elements of $\binom{T}{r}$ are coloured with the $i$th colour.*

**Proof** We have proved Ramsey's theorem for 2 colours. Now consider the case of $s$ colours. We show that

$$N(q_1, q_2, \ldots, q_s; r) \leq N(Q_1, Q_2; r)$$

where

$$\begin{aligned} Q_1 &= N(q_1, q_2, \ldots, q_{\lfloor s/2 \rfloor}; r) \\ Q_2 &= N(q_{\lfloor s/2 \rfloor + 1}, q_{\lfloor s/2 \rfloor + 2}, \ldots, q_s; r) \end{aligned}$$

Let $n = N(Q_1, Q_2; r)$ and assume we are given an $s$-colouring of $\binom{[n]}{r}$. First temporarily re-colour Red, any $r$-set coloured with $i \leq \lfloor s/2 \rfloor$ and re-colour Blue any $r$-set coloured with $i > \lfloor s/2 \rfloor$.

Then either (a) there exists a $Q_1$-subset $A$ of $[n]$ with $\binom{A}{r}$ coloured Red or (b) there exists a $Q_2$-subset $B$ of $[n]$ with $\binom{A}{r}$ coloured Blue.

W.l.o.g. assume the first case. Now replace the colours of the $r$-sets of $A$ by there original colours. We have a $\lfloor s/2 \rfloor$-colouring of $\binom{A}{r}$. Since $|A| = N(q_1, q_2, \ldots, q_{\lfloor s/2 \rfloor}; r)$ there must exist some $i \leq \lfloor s/2 \rfloor$ and a $q_i$-subset $S$ of $A$ such that all of $\binom{S}{r}$ has colour $i$. □

We now prove a lower bound on $R(k, k)$.

**Theorem 2.** *Assume that $k \geq 3$. Then*

$$R(k, k) \geq 2^{k/2}.$$

**Proof** We must prove that if $n \leq 2^{k/2}$ then there exists a Red-Blue colouring of the edges of $K_n$ which contains no Red $k$-clique and no Blue $k$-clique. We can assume $k \geq 4$ since we know $R(3, 3) = 6$.

We show that this is true with positive probability in a *random* Red-Blue colouring. So let $\Omega$ be the set of all Red-Blue edge colourings of $K_n$ with uniform distribution. Equivalently we independently colour each edge Red with probability 1/2 and Blue with probability 1/2.

Let
$\mathcal{E}_R$ be the event: {There is a Red $k$-clique} and
$\mathcal{E}_B$ be the event: {There is a Blue $k$-clique}.

We show

$$\mathbf{Pr}(\mathcal{E}_R \cup \mathcal{E}_B) < 1.$$

Let $C_1, C_2, \ldots, C_N$, $N = \binom{n}{k}$ be the vertices of the $N$ $k$-cliques of $K_n$.
Let $\mathcal{E}_{R,j}$ be the event: $\{C_j$ is Red$\}$.

Now

$$
\begin{aligned}
\mathbf{Pr}(\mathcal{E}_R \cup \mathcal{E}_B) &\leq \mathbf{Pr}(\mathcal{E}_R) + \mathbf{Pr}(\mathcal{E}_B) \\
&= 2\mathbf{Pr}(\mathcal{E}_R) \\
&= 2\mathbf{Pr}\left(\bigcup_{j=1}^{N} \mathcal{E}_{R,j}\right) \\
&\leq 2\sum_{j=1}^{N} \mathbf{Pr}(\mathcal{E}_{R,j}) \\
&= 2\sum_{j=1}^{N} \left(\frac{1}{2}\right)^{\binom{k}{2}} \\
&= 2\binom{n}{k}\left(\frac{1}{2}\right)^{\binom{k}{2}} \\
&\leq 2\frac{n^k}{k!}\left(\frac{1}{2}\right)^{\binom{k}{2}} \\
&\leq 2\frac{2^{k^2/2}}{k!}\left(\frac{1}{2}\right)^{\binom{k}{2}} \\
&= \frac{2^{1+k/2}}{k!} \\
&< 1.
\end{aligned}
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$\square$