Class 07

Colorings of graphs and Ramsey's Theorem

Ramsey's Theorem for graphs:

Theorem 1. *Let k, ℓ be given. There exists $R(k,\ell)$ such that if the edges of K_n are coloured Red or Blue and $n \geq R(k,\ell)$ then either (i) $\exists K_k \subseteq K_n$ all of whose edges are Red or (ii) $\exists K_\ell \subseteq K_n$ all of whose edges are Blue.*

Proof We will show that for $k, \ell \geq 3$ that

$$R(k,\ell) \leq R(k-1,\ell) + R(k,\ell-1). \tag{1}$$

Let $n = R(k-1,\ell) + R(k,\ell-1)$ be given and let $V_R = \{x : \ (x,n) \text{ is coloured Red}\}$ and let $V_B = \{x : \ (x,n) \text{ is coloured Blue}\}$. Then

$$|V_R| + |V_B| = R(k-1,\ell) + R(k,\ell-1) - 1$$

and so

$$(i)\ |V_R| \geq R(k-1,\ell) \text{ or } (ii)\ |V_B| \geq R(k,\ell-1).$$

Assume w.l.o.g. that $|V_R| \geq R(k-1,\ell)$. Then either

V_R contains a Red clique S of size $k-1$ and so $S \cup \{n\}$ is a Red clique of size k

or

V_R contains a Blue clique T of size ℓ. □

Corollary 2.

$$R(k,\ell) \leq \binom{k+\ell-2}{k-1}.$$

Proof By induction on $k+\ell$. True for $k+\ell \leq 5$.

$$\begin{aligned} R(k,\ell) &\leq R(k-1,\ell) + R(k,\ell-1) \\ &\leq \binom{k+\ell-3}{k-2} + \binom{k+\ell-3}{k-1} \qquad \text{induction} \\ &= \binom{k+\ell-2}{k-1}. \end{aligned}$$

□

Corollary 3.

$$R(k,k) \leq \binom{2k-2}{k} \leq 4^k.$$

Ramsey's Theorem in general

Theorem 4. *Let $r, s \geq 1$, $q_i \geq r$, $1 \leq i \leq s$ be given. Then there exists $N = N(q_1, q_2, \ldots, q_s; r)$ with the following property: Suppose that S is a set with $n \geq N$ elements. Let each of the elements of $\binom{S}{r}$ be given one of s colours. (***Note that the elements of $\binom{S}{r}$ are the r-subsets of S**).*

Then there exists i and a q_i-subset T of S such that all of the elements of $\binom{T}{r}$ are coloured with the ith colour.

Proof First assume that $s = 2$ i.e. two colours, Red, Blue.

Note that

$$\begin{aligned} (a)\ N(p,q;1) &= p+q-1 \\ (b)\ N(p,r;r) &= p(\geq r) \\ N(r,q;r) &= q(\geq r) \end{aligned}$$

We proceed by induction on r. It is true for $r = 1$ and so assume $r \geq 2$ and it is true for $r - 1$ and arbitrary p, q.

Now we further proceed by induction on $p+q$. It is true for $p+q = 2r$ and so assume it is true for r and p', q' with $p' + q' < p + q$.

let

$$\begin{aligned} p_1 &= N(p-1,q;r) \\ p_2 &= N(p,q-1;r) \end{aligned}$$

These exist by induction.

Now we prove that

$$N(p,q;r) \leq 1 + N(p_1,q_1;r-1)$$

where the RHS exists by induction.

Now coulor $\binom{[n]}{r}$ with 2 colours. call this colouring σ. From this we define a colouring τ of $\binom{[n-1]}{r-1}$ as follows: If $X \subseteq [n-1]$ then give it the colour of $X \cup \{n\}$ under σ.

Now either (i) there exists $A \subseteq [n-1]$, $|A| = p_1$ such that (under τ) all members of $\binom{A}{r-1}$ are Red or (ii) there exists $B \subseteq [n-1]$, $|A| = q_1$ such that (under τ) all members of $\binom{B}{r-1}$ are Blue.

Assume w.l.o.g. that (i) holds.

$$|A| = p_1 = N(p-1,q;r).$$

Then either

(a) $\exists B \subseteq A$ such that $|B| = q$ and under σ all of $\binom{B}{r}$ is Blue – done, or

(b) $\exists A' \subseteq A$ such that $|A'| = p-1$ and all of $\binom{A'}{r}$ is Red. But then all of $\binom{A' \cup \{n\}}{r}$ is Red. If $X \subseteq A'$, $|X| = r-1$ then τ colours X Red, since $A' \subseteq A$. But then σ will colour $X \cup \{n\}$ Red.