**Class 06**

Colorings of graphs and Ramsey's Theorem

**Lemma 1.** *If we colour the edges of $K_6$ with 2 colours, Red and Blue then either there is a Red triangle or there is a Blue triangle.*

**Proof** At least 3 of the edges incident with vertex 1 have the same colour. Assume without loss of generality that these edges are coloured Red. Assume further that these edges are $(1,2),(1,3),(1,4)$. Now consider the triangle $T=(2,3,4)$. If any of the edges of $T$ are Red then we have a Red triangle, otherwise $T$ is Blue. □

Ramsey theory is a far reaching generalisation of the above simple lemma.

**Theorem 1.** *The edges of $K_n$ are coloured Red or Blue. $r_i$ is the number of Red edges incident with vertex $i$. Let $\Delta$ denote the number of monochromatic triangles. Then*

$$\Delta = \binom{n}{3} - \frac{1}{2}\sum_{i=1}^{n} r_i(n-1-r_i). \tag{1}$$

**Proof** Each non-monochromatic triangle contains 2 adjacent pairs of edges of different colour. The number of pairs of edges of a different colour incident with vertex $i$ is $r_i(n-1-r_i)$. Thus the sum in (1) is the total number of differently coloured adjacent pairs. We obtain the number of non-monochromatic triangles by dividing by 2. □

**Corollary 2.**

$$\Delta \geq \binom{n}{3} - \left\lfloor \frac{n}{2}\left\lfloor \left(\frac{n-1}{2}\right)^2 \right\rfloor \right\rfloor.$$

**Proof** $\Delta$ is minimised if $r_i = \frac{n-1}{2}\forall i$ ($n$ odd) and if $r_i = \frac{n}{2}\forall i$ ($n$ even). Thus

$n$ **odd**

$$\Delta \geq \binom{n}{3} - \frac{n}{2}\left(\frac{n-1}{2}\right)^2 = \binom{n}{3} - \left\lfloor \frac{n}{2}\left\lfloor \left(\frac{n-1}{2}\right)^2 \right\rfloor \right\rfloor.$$

$n$ **even**

$$\Delta \geq \binom{n}{3} - \left\lfloor \frac{n}{2}\frac{n}{2}\left(\frac{n}{2}-1\right)\right\rfloor = \binom{n}{3} - \left\lfloor \frac{n}{2}\left\lfloor \left(\frac{n-1}{2}\right)^2 \right\rfloor \right\rfloor$$

since $\frac{n}{2}\left(\frac{n}{2}-1\right) = \left\lfloor \left(\frac{n-1}{2}\right)^2 \right\rfloor$.

□

Examples: $n=6$ $\Delta \geq 2$. $n=7$ $\Delta \geq 4$.

Ramsey's Theorem for graphs:

**Theorem 3.** *Let $k,\ell$ be given. There exists $R(k,\ell)$ such that if the edges of $K_n$ are coloured Red or Blue and $n \geq R(k,\ell)$ then either (i) $\exists K_k \subseteq K_n$ all of whose edges are Red or (ii) $\exists K_\ell \subseteq K_n$ all of whose edges are Blue.*

Simple cases:

- $R(k,1) = R(1,k) = 1$.
- $R(k,2) = R(2,k) = k$.
- $R(3,3) = 6$. (Here we need the fact that $K_5$ can be partitioned into 2 $C_5$'s.