Class 05

Colorings of graphs and Ramsey's Theorem

A **proper colouring** of the vertices of graph G is a map $f: V \to C$ such that v, w adjacent implies $f(v) \neq f(w)$. If $|C| = k$ then f is a k-colouring.

The **Chromatic number** $\chi(G) = \min\{k : \ G \text{ has a } k\text{-colouring}\}$.

Examples:
$\chi(K_n) = n$, $\chi(C_{2k}) = 2$ and $\chi(C_{2k+1}) = 3$.

Let $\Delta(G)$ be the maximum degree of G.

Lemma 1. *G is $(\Delta + 1)$-colourable.*

Proof Let the vertices of G be $v_1, v_2, \ldots, v_n$.

Algorithm: For $i = 1, 2, \ldots, n$, colour v_i with a colour not used in $\Gamma(v_i) \cap \{v_1, v_2, \ldots, v_{i-1}\}$. This is always possible since $|\Gamma(v_i) \cap \{v_1, v_2, \ldots, v_{i-1}\}| \leq \Delta$. □

Theorem 1. *Suppose that $d \geq 3$ and $\Delta(G) \leq d$ and that K_{d+1} is not a subgraph of G. Then $\chi(G) \leq d$.*

Proof We use induction on $|V|$. We can assume that G is connected, since we can apply the theorem (inductively) We can also assume that $\Delta(G) = d$, for otherwise we can use Lemma 1.

1. $\exists v : \ G - v$ is not connected. Suppose that the components of $G - v$ are $C_1, C_2, \ldots, C_k$. We can by induction colour $D_i = C_i + v$ for $i = 1, 2, \ldots, k$ using d colours. We can further assume that colour 1 is used for v in all of these colourings. But then we can put the colourings together to make a d-colouring of G.

2. $\exists v_1, v_2 : \ G - \{v_1, v_2\}$ is not connected. Add edge (v_1, v_2) if necessary. Suppose that the components of $G - \{v_1, v_2\}$ are $C_1, C_2, \ldots, C_k$. We can by induction colour $D_i = C_i + \{v_1, v_2\}$ for $i = 1, 2, \ldots, k$ using d colours. Since v_1, v_2 are adjacent, we can assume that v_1 has colour 1 and v_2 has colour 2 in all of these colourings. But then we can put the colourings together to make a d-colouring of G.

3. We are now left with the case where we can assume that for any set $X \neq V$ of ≥ 3 vertices, there exist $x, y, z \in X$ such that each of x, y, z is adjacent to some vertex in $V \setminus X$. Since $G \neq K_{d+1}$ there are vertices x_1, x_{n-1}, x_n such that x_1 is adjacent to x_{n-1}, x_n and x_{n-1} is not adjacent to x_n.

   Now order the vertices as $x_1, x_2, \ldots, x_n$ so that each $x_k, k \geq 2$ is adjacent to at leat one $x_i, i < k$. This can be done: If we have chosen $x_, x_, \ldots, x_{k-1}$ then we can choose x_k to be any vertex in $X = V \setminus \{x_1, x_2, \ldots, x_{k-1}\}$ other than x_{n-1}, x_n which is adjacent to something in $V \setminus X$.

   Now assign x_{n-1}, x_n the same colour. We can colour $x_{n-2}, x_{n-3}, \ldots, x_2$ in this order so that no x_k has the same colour as a neighbour x_l with $l > k$. This is because x_k has most $d - 1$ neighbours x_l with $l > k$.

   Finally, the neighbourhood of x_1 uses at most $d - 1$ colours, since the colour of x_{n-1}, x_n has been used at least twice.

□