Class 04

Theorem 1. *There are n^{n-2} different trees with vertex set $[n]$.*

Proof Let $\mathcal{T}$ be the set of trees with vertex set $[n]$. We define a map $\mathcal{P} : \mathcal{T} \to [n]^{n-2}$ such that $\mathcal{P}$ is a bijection. Then $|\mathcal{T}| = |[n]^{n-2}| = n^{n-2}$.

We define a sequence $(x_i, y_i, T_i), i = 1, 2, \ldots, n-1$ where $T_1 = T$ and x_i is the least valued vertex of degree 1 in T_i and y_i is the **unique** neighbour of x_i in T_i.

$$\mathcal{P}(T) = y_1 y_2 \cdots y_{n-2}.$$

We need to show that we can recover T from $\mathcal{P}(T)$ i.e. $\mathcal{P}$ is 1-1 and onto. We can prove this by induction on n. Observe first that vertex v occurs $deg_T(v) - 1$ times among $y_1, y_2, \ldots, y_{n-2}$. This is because it occurs $deg_T(v)$ times among $x_1, y_1, x_2, y_2, \ldots, x_{n-1}, y_{n-1} = n$ and exactly once among $x_1, x_2, \ldots, x_{n-1}, y_{n-1}$.

Thus x_1 is the least element of V which does not appear in $y_1, y_2, \ldots, y_{n-1}$ and then T is T_1 plus vertex x_1 and edge (x_1, y_1). The rest of the code defines T_1, by induction. (Note that T_1 is a tree on $[n] \setminus \{x_1\}$ and so we really need to define the code for trees with vertices in any ordered set). $\square$