Class 03

Trees

A **tree** is a connected graph without any cycles.

Lemma 1. *A finite tree T with $n \geq 2$ vertices has at least 2 vertices of degree 1.*

Proof Choose $x_1 \in V(T)$ arbitrarily. Choose a neighbour x_2 of x_1. Suppose then that we have chosen distinct vertices $x_1, x_2, \ldots, x_i, i \geq 2$.

Case 1 $deg(x_i) = 1$ – done!

Case 2 If $deg(x_i) \geq 2$ then there exists $x \neq x_{i-1}$ such that x is adjacent to x_i. Now $x \notin \{x_1, x_2, \ldots, x_{i-1}\}$ for if $x = x_k$ then T contains the cycle $x_k, \ldots, x_i, x, x_k$. so we take $x_{i+1} = x$ and replace i by $i + 1$.

Since T is finite we must eventually end up in Case 1. This gives us one vertex of degree 1.

Now putting $y_j = x_{i-j+1}$ for $1 \leq j \leq i$ we continue the argument with $y_1, y_2, \ldots, y_i$ in place of $x_1, x_2, \ldots, x_i$ (we are now in effect growing our path from x_1 avoiding previously seen vertices.). In this way we find a second vertex of degree 1. □

Lemma 2. *Suppose that $T = (V, E)$ is a tree with n vertices. Suppose that $x \in V$ has degree 1 and that $e = \{x, y\}$ is the unique edge incident with x. Then $T' = (V \setminus \{x\}, E \setminus \{e\})$ is also a tree.*

Proof T' has no cycles since $T' \subseteq T$. We must show that it is connected. Let $a, b \neq x$ be vertices of T'. There is a path $P = (a = x_0, x_1, \ldots, x_k = b)$ from a to b in T. Let this be as short as possible. We claim that P is a path in T'. If not, then there exists i such that $x_i = x$. But then we must have $x_{i-1} = x_{i+1} = y$ and we get a shorter path by simply removing x_{i-1}, x_i, contradiction. □

Corollary 1. *A tree T with n vertices has $n - 1$ edges.*

Proof By induction on n. If $n = 1$ then T has no edges.

Assume the result for some n and let T be a tree with $n + 1$ vertices. Choose x, e, T' as in Lemma 2. Then T' has n vertices and so by induction it has $n - 1$ edges. But then T has $n - 1 = 1$ edges. □

Theorem 2. *There are n^{n-2} different trees with vertex set $[n]$.*

Proof Let $\mathcal{T}$ be the set of trees with vertex set $[n]$. We define a map $\mathcal{P} : \mathcal{T} \to [n]^{n-2}$ such that $\mathcal{P}$ is a bijection. Then $|\mathcal{T}| = |[n]^{n-2}| = n^{n-2}$.

We define a sequence $(x_i, y_i, T_i), i = 1, 2, \ldots, n - 1$ where $T_1 = T$ and x_i is the least valued vertex of degree 1 in T_i and y_i is the **unique** neighbour of x_i in T_i.

$$\mathcal{P}(T) = y_1 y_2 \cdots y_{n-2}.$$

Continued in next class.