Class 02

Eulerian Graphs

An *Eulerian cycle* of a graph $G = (V, E)$ ia **closed** walk which uses each edge $e \in E$ **exactly** once.

Theorem 1. *A connected graph with at least one edge is Eulerian i.e. has an Eulerian cycle, iff it has no vertex of odd degree.*

Proof Suppose $W = (v_0, e_1, v_1, e_2, v_2, \ldots, e_m, v_m = v_0)$, $m = |E|$ is an Eulerian cycle. $v \in V$. Whenever W visits v it enters through a new edge $e - i$ and leaves through a new edge e_{i-1}. Thus each visit requires 2 new edges. Thus the degree of v is even.

The converse is proved by induction on $|E|$. The result is true for $|E| \leq 3$. The only possible graphs are (i) a loop, (ii) a pair of parallel edges or (iii) a triangle.

Assume $|E| \geq 4$. G is not a tree (see below), since it has no vertex of degree 1. Therefore it contains a cycle C. Delete the edges of C. The remaining graph has components $K_1, K_2, \ldots, K_s$ where $K_1, K_2, \ldots, K_r$ are the non-trivial components.

Each K_i is connected and is of even degree – deleting the edges of C removes 0 or 2 edges incident with a given $v \in V$. Also, each K_i has strictly less than $|E|$ edges. So, by induction, each K_i, which is not an isolated vertex has an Eulerian cycle, C_i say.

We create an Eulerian cycle of G as follows: let $C = (v_1, e_1, v_2, \ldots, v_s, e_s, v_1)$. Let v_{i_t} be the first vertex of C which is in K_t. Assume w.l.o.g. that $i_1 < i_2 < \cdots < i_r$.

$W = (v_1, e_1, v_2, \ldots, v_{i_1}, C_1, e_{i_1},, \ldots, v_{i_2}, C_2, e_{i_2}, \ldots, v_{i_r}, C_r, e_{i_r}, \ldots, v_1)$

is an Eulerian cycle of G.

A **tree** is a connected graph without any cycles.

Theorem 2. *A tree T with $n \geq 2$ vertices has at least 2 vertices of degree 1.*