Class 01

A graph $G = (V, E)$ comprises a set of vertices V and a set of 2-element subsets of V called the vertices. A graph is usually drawn as points for the vertices plus a line joining a to b for every edge $\{a, b\} \in E$. If $\{a, b\} \in E$ then we say that a is **adjacent** to b. This defines a **simple** graph.

If E is a multi-set then we have a **multi-graph** and in this case we allow **loops** which are edges of the form $\{x, x\}$.

A graph is **planar** if it can be drawn on the plane in such a way that no edges meet, except at common vertices.

The **complete graph** K_n has vertex set $[n]$ and all possible $\binom{n}{2}$ possible edges. A graph is **bipartite** if its vertex set can be partitioned into 2 sets V_1, V_2 such that every edge *joins* a vertex in V_1 to a vertex in V_2. The **complete bipartite graph** $K_{m,n}$ has vertex bipartition $[m]$ and $[n]$ and all possible mn edges.

K_5 and $K_{3,3}$ are both non-planar.

A **digraph** or **directed graph** has vertices V and now an edge is an ordered pair of vertices. When drawing a digraph we add an arrow to the edge (a, b) in the direction of b.

Two graps $G_i = (V_i, E_i), i = 1, 2$ are **isomorphic** if there is a bijection $f : V_1 \to V_2$ such that $\{a, b\} \in E_1$ iff $\{f(a), f(b)\} \in V_2$.

The **neighbourhood** $\Gamma(x)$ of vertex x in a graph G is the set of vertices which are adjacent to x. The degree $deg(x)$ is equal to $|\Gamma(x)|$.

Theorem 1. *A finite graph G has an even number of vertices of odd degree.*

Proof Consider the $V \times E$ matrix A wherw $A(v, e) = 1$ iff $v \in e$. Then

$$\#1's \text{ in } A = 2|E| = \sum_{x \in V} deg(x).$$

(2 1's per column gives $2|E|$ and row x has $deg(x)$ 1's for $x \in V$.) □

A **walk** in G is a sequence $x_0, e_1, x_2, \ldots, e_k, x_k$ where e_i is an edge joining x_{i-1} to x_i. The walk has **length** k, equals number of edges.

If the edges are distinct then we call the walk a **path** — non-standard notation. If the vertices are distinct then we have a **simple path**.

A **simple closed path** is called a **polygon** — **cycle** is more standard.

Distance $d(a, b)$ is the length of the shortest walk from vertex a to vertex b.

A graph is **connected** if every pair of vertices are joined by a path. Otherwise, G is the union of a number of connected subgraphs called **connected components**.