ON THE TOPOLOGY OF RANDOM COMPLEXES BUILT OVER STATIONARY POINT PROCESSES

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There has been considerable recent interest, primarily motivated by problems in applied algebraic topology, in the homology of random simplicial complexes. We consider the scenario in which the vertices of the simplices are the points of a random point process in $\mathbb{R}^d$, and the edges and faces are determined according to some deterministic rule, typically leading to Čech and Vietoris-Rips complexes. In particular, we obtain results about homology, as measured via the growth of Betti numbers, when the vertices are the points of a general stationary point process. This significantly extends earlier results in which the points were either iid observations or the points of a Poisson process. In dealing with general point processes, in which the points exhibit dependence such as attraction, or repulsion, we find phenomena quantitatively different from those observed in the iid and Poisson cases. From the point of view of topological data analysis, our results seriously impact on considerations of model (non) robustness for statistical inference.

1. Introduction. There has been considerable recent interest, primarily motivated by problems in applied algebraic topology, in the homology of random simplicial complexes. Two main scenarios have been considered. In the more common, the vertices of the simplices are a random point set, and the edges and faces are determined according to some deterministic rule, typically related to the distance between pairs, or general subsets, of vertices. This has lead, for example, to the random Čech and Vietoris-Rips complexes studied in papers such as [1, 18, 19].

Another approach has been to consider random subsets of complete graphs, leading to a number of papers dealing with the homology of Erdös-Rényi clique complexes, as in, for example, [12, 17, 22, 25].

The current paper is concerned with the first of these approaches, although with a novel and - from the point of view of both theory and ap-

plications – important change of emphasis. Previous papers on simplicial complexes built over random point sets have always assumed that the points were either independent, identically distributed (iid) observations from some underlying distribution on $\mathbb{R}^d$, or points of a (typically non-homogeneous) Poisson point process. Our aim in this paper is to investigate situations in which the points are chosen from a general point process, in which the points exhibit dependence such as attraction or repulsion. From the point of view of topological data analysis (TDA) our results, which show that local dependencies can have a major affect on the growth rates of topological quantifiers such as Betti numbers, impact on considerations of model (non) robustness for statistical inference in TDA. We shall not address these issues here, however, beyond a few comments in Section 1.3 below.

To start being a little more specific, given a point process (i.e. locally finite random counting measure) Φ on $\mathbb{R}^d$, recall that the random geometric graph $G(\Phi, r)$, for $r > 0$, is defined as the graph with vertex set Φ and (undirected) edge set $\{(X, Y) \in \Phi^2 : \|X - Y\| \leq r\}$. The properties of random geometric graphs when Φ is a Poisson point process or a point process of iid points have been analysed in detail (cf. [30]), and recently interest has turned to the richer topic of random simplicial complexes built over these point sets. We shall be concerned with Čech and Vietoris-Rips complexes. Let $B_x(\varepsilon)$ denote the ball of radius ε around x, and $\Phi = \{x_1, x_2, \ldots\}$ be a collection of points in $\mathbb{R}^d$.

DEFINITION 1.1 (Čech complexes).   *The abstract simplicial complex $C(\Phi, \varepsilon)$ constructed according to the following rules is called the Čech complex associated to Φ and ε.*

1. *The 0-simplices of $C(\Phi, \varepsilon)$ are the points in Φ,*
2. *An n-simplex, or n-dimensional 'face', $\sigma = [x_{i_0}, \ldots, x_{i_n}]$ is in $C(\Phi, \varepsilon)$ if $\bigcap_{k=0}^{n} B_{x_{i_k}}(\varepsilon/2) \neq \emptyset$,*

DEFINITION 1.2 (Vietoris-Rips complexes).   *The abstract simplicial complex $R(\Phi, \varepsilon)$ constructed according to the following rules is called the Vietoris-Rips complex associated to Φ and ε.*

1. *The 0-simplices of $R(\Phi, \varepsilon)$ are the points in Φ.*
2. *An n-simplex, or n-dimensional 'face', $\sigma = [x_{i_0}, \ldots, x_{i_n}]$ is in $R(\Phi, \varepsilon)$ if $B_{x_{i_k}}(\varepsilon/2) \cap B_{x_{i_m}}(\varepsilon/2) \neq \emptyset$ for every $0 \leq k < m \leq n$.*

More information on these complexes will be given in Section 4.1 when it is needed. Both of these (related) complexes are important in their own right,

with the Čech complex being of particular interest since it is known to be homotopy equivalent to the random Boolean set $\bigcup_{x \in \Phi} B_x(\varepsilon)$ which appears in integral geometry (e.g. [31]) and continuum percolation (e.g. [24]). We shall concentrate in this paper on the ranks of the the homology groups – i.e. the Betti numbers – of these complexes in the random scenario. At a heuristic level, the k-th Betti number, β_k, measures the number of k-dimensional cycles, or "holes", in the complex.

A complementary approach to studying the topological structure of simplicial complexes is via (non-smooth) Morse theory, and here results for Poisson process generated complexes are given in [7]. In fact, the local structure of Morse critical points is often more amenable to computation than the global structure of the Betti numbers. Thus we shall also take this route in parts of this paper.

There are some recurring themes and techniques in the analysis of Betti numbers and Morse critical points, which are intimately related to the subgraph and component counts of the corresponding random geometric graph. Thus, from the purely technical side, much of this paper will be concerned with the intrinsically interesting task of extending the results of [30, Chapter 3] on subgraph and component counts of Poisson point processes to more general stationary point processes. Subgraph counts of a random geometric graph are an example of U-statistics of point processes. Hence, apart from their applications in this article, our techniques to study subgraph counts of random geometric graph over general stationary point processes could be useful to derive asymptotics for many other translation and scale invariant U-statistics of point processes (For example, the number of k-simplices in a Čech or Vietoris-Rips complexes). Also, the results on subgraph counts are used to derive results about clique numbers, maximum degree and chromatic number of the random geometric graph on Poisson or i.i.d. point process ([30, Chapter 6]) and with a similar approach, our results can be used to derive asymptotics for clique numbers, maximum degree and chromatic number of random geometric graphs over general stationary point processes (see [4, Section 4.3.1]). This will take up all of Section 3, the longest section of the paper. From these results, we shall be able to extract results about Betti numbers (via combinatorial topology) as well as the numbers of Morse critical points. In formulating results, we shall relate the topological features of the random simplicial complexes to known, inherent properties of the underlying point processes, including joint densities, void probabilities or Palm void probabilities. The first two of these properties, along with association properties, are known to be useful in studying measures of clustering, and their impact on percolation of random geometric graphs was studied in [5].

Since our asymptotic results help quantify the impact of clustering measures such as sub-Poisson and negative association on topological features of point processes, they provide additional applications of the tools of [6] as measures of clustering.

A sampler of some of our main results follows a little necessary notation.

1.1. *Some notation.* We use $|.|$ to denote Lebesgue measure and $\|.\|$ for the Euclidean norm on $\mathbb{R}^d$. Depending on context, $|.|$ will also denote the cardinality of a set. As above, we denote the ball of radius r centred at $x \in \mathbb{R}^d$ by $B_x(r)$. For $\mathbf{x} = (x_1, \ldots, x_k) \in \mathbb{R}^{dk}$, let $B_{\mathbf{x}}(r) = \bigcup_{i=1}^k B_{x_i}(r)$, $h(\mathbf{x}) = h(x_1, \ldots, x_k)$ for $h : \mathbb{R}^{dk} \to \mathbb{R}$ and $d\mathbf{x} = dx_1 \ldots dx_k$. Let $\mathbf{1} = (1, \ldots, 1)$. We also use the standard Bachman-Landau notation for asymptotics[1], and say that a sequence of events A_n, $n \geq 1$ occurs *with high probability (whp)* if $\mathbb{P}\{A_n\} \to 1$ as $n \to \infty$.

1.2. *A result sampler.* We shall now describe, without (sometimes important) precise technical conditions, some of our main results. Full details are given in the main body of the paper. We start with Φ, a unit intensity, stationary point process on $\mathbb{R}^d$, and set [2]

$$(1.1) \qquad \Phi_n \ = \ \Phi \cap \left[\frac{-n^{1/d}}{2}, \ \frac{n^{1/d}}{2}\right]^d.$$

Let

$$\beta_k\left(C(\Phi_n, r)\right), \qquad \beta_k\left(R(\Phi_n, r)\right),$$

respectively, denote the k-th Betti numbers of the Čech and Vietoris-Rips complexes based on Φ_n.

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[1]That is, for sequences a_n and b_n of positive numbers, we write

$$
\begin{aligned}
a_n = o(b_n) &\iff && \text{for any } c > 0 \text{ there is a } n_0 \text{ such that } a_n < cb_n \text{ for all } n > n_0, \\
a_n = O(b_n) &\iff && \text{there exists a } c > 0 \text{ and a } n_0 \text{ such that } a_n < cb_n \text{ for all } n > n_0, \\
a_n = \omega(b_n) &\iff && \text{for any } c > 0 \text{ there is a } n_0 \text{ such that } a_n > cb_n \text{ for all } n > n_0, \\
a_n = \Omega(b_n) &\iff && \text{there exists a } c > 0 \text{ and a } n_0 \text{ such that } a_n > cb_n \text{ for all } n > n_0, \\
a_n = \Theta(b_n) &\iff && a_n = O(b_n) \text{ and } a_n = \Omega(b_n)
\end{aligned}
$$

[2]Note that our basic setup is a little different different to that of all the earlier papers mentioned above. To compare our results with existing ones on Poisson or iid point processes, note that r_n^d in our results typically corresponds to nr_n^d elsewhere. For a general (non-Poisson) point process, (1.1) provides a more natural setting.

In addition, let $C_k(\Phi_n, r)$ denote the set of Morse critical points of index k for the distance function

$$d_n(x) = \min_{X \in \Phi_n} \|x - X\|,$$

and set

$$N_k(\Phi_n, r) = |\{c \in C_k(\Phi_n, r) : d_n(c) \le r\}|.$$

This paper is concerned with the behavior, as $n \to \infty$, of $\beta_k(C(\Phi_n, r_n))$, $\beta_k(R(\Phi_n, r_n))$, $N_k(\Phi_n, r_n)$ and $\chi(C(\Phi_n, r_n))$, where χ denotes the Euler characteristic. In particular, we shall provide closed form expressions for the asymptotic, normalised, first moments of these variables, along with bounds for second moments for most of them.

Throughout the remainder of this subsection we shall assume that Φ is stationary, unit mean, and negatively associated (defined rigorously in Section 2.2). Additional side conditions may also need to hold, but we shall not state them here. Two simple examples for which everything works are provided by the Ginibre point process and the simple perturbed lattice. Many of the results hold for various other sub-classes of point processes as well, but our non-specific blanket assumptions allow for ease of exposition. We divide the results into three classes, depending on the behavior of r_n.

I. SPARSE REGIME: $r_n \to 0$. Note that since the points of Φ only generate edges and faces of the Čech and Vietoris-Rips complexes $C(\Phi_n, r)$ and $R(\Phi_n, r)$ when they are distance less than r apart, and since Φ has, on, average, only one point per unit cube, if r is small we expect that both of these complexes will be made up primarily of the isolated points of Φ. We describe this fact by calling this the 'sparse' regime.

In this setting,

$$\mathbb{E}\{\beta_0(C(\Phi_n, r_n)\} = \mathbb{E}\{\beta_0(R(\Phi_n, r_n)\} = \Theta(n)$$

and for $k \ge 1$, there exist functions $f^k(r) \equiv 1$ or $f^k(r) \to 0$, as $r \to 0$, depending on the precise distribution of Φ and on the index k, such that

$$(1.2) \qquad \begin{aligned} \mathbb{E}\{\beta_k(C(\Phi_n, r_n)\} &= \Theta(nr_n^{d(k+1)} f^{k+2}(r_n)), \\ \mathbb{E}\{\beta_k(R(\Phi_n, r_n)\} &= \Theta(nr_n^{d(2k+1)} f^{2k+2}(r_n)), \\ \mathbb{E}\{N_k(\Phi_n, r_n)\} &= \Theta(nr_n^{dk} f^{k+1}(r_n)), \end{aligned}$$

and $\mathsf{Var}(N_k(\Phi_n, r_n)) = O(\mathbb{E}\{N_k(\Phi_n, r_n)\})$, where $\mathsf{Var}(X)$ is the variance of X. In addition, $\mathbb{E}\{n^{-1}\chi(C(\Phi_n, r_n))\} \to 1$.

In the classical Poisson case, studied in the references given above, it is known that the same results hold with $f \equiv 1$.

Using stochastic ordering techniques, we shall also show that clustering of point processes increases the functions $f^k(r)$ and consequently the mean of the β_k and N_k as well. Also, we know that for the Ginibre point process and for the zeroes of Gaussian entire functions, $f^k(r) = r^{k(k-1)}$. Thus, there is a systematic difference between the scaling limits for Poisson and at least some negatively associated point processes.

II. THERMODYNAMIC REGIME: $r_n^d \to \beta \in (0, \infty)$. In this regime an edge between two points in Φ, which are, in a rough sense, an average distance of one unit apart, will be formed if they manage to get within a distance $\beta^{1/d}$ of one another. Since, in most scenarios, there should be a reasonable probability of this happening, we expect to see quite a few edges and, in fact, simplices and homologies of all dimensions. Indeed, this is the case, and the main result in this regime is that topological complexity grows at a rate proportional to the number of points, in the sense that

$$\mathbb{E}\{\beta_k(C(\Phi_n, r_n)\} = \Theta(n),$$

with identical results for $\mathbb{E}\{\beta_k(R(\Phi_n, r_n))\}$ and $\mathbb{E}\{N_k(\Phi_n, r_n)\}$. In addition, $\mathsf{Var}(N_k(\Phi_n, r_n)) = O(\mathbb{E}\{N_k(\Phi_n, r_n)\})$ and

$$\mathbb{E}\{n^{-1}\chi(C(\Phi_n, r_n))\} \to 1 + \sum_{k=1}^{d}(-1)^k \nu_k(\Phi, \beta),$$

where the $\nu_k(\Phi, \beta)$ are defined in Theorem 5.2. Since there is no appearance in these results of an analogue to the f of (1.2), the normalisations here have the same orders as in the Poisson and iid cases.

III. CONNECTIVITY REGIME: $r_n^d = \Theta(\log n)$. Clearly, if r_n is large enough, there comes a point (which we call the *contractibility radius*) beyond which all the points of Φ_n will connect with the others and both the Čech and Vietoris-Rips simplices will become trivial, in the sense that the graph over the points will be totally connected, and so the various simplicial complexes will be contractible to a single point. (This is certainly the case if $r_n = \sqrt{d}n^{1/d}$.) The question, then, is "how large is large enough".

It turns out that in the current scenario of negative association there exist case dependent constants C such that for, $r_n \geq C(\log n)^{1/d}$, $C(\Phi_n, r_n)$ is contractible whp as $n \to \infty$. In the specific cases of the Ginibre process or zeroes of Gaussian entire functions, this happens earlier, and $r_n = \Theta((\log n)^{1/4})$ is the radius for contractibility of the Čech complex. As a trivial corollary, it

follows that, whp, $\chi(C(\Phi_n, r_n)) = 1$ when r_n is the radius of contractibility. Further, for the Ginibre process, $r_n = \Theta((\log n)^{1/4})$ is also the critical radius for k-connectedness of the Vietoris-Rips complex.

1.3. *Some implications for topological data analysis.* Perhaps the core tool of TDA is persistent homology, as visualised through barcodes and persistence diagrams (cf. [10, 13, 16, 35]). While here is not the place to go into the details of persistent homology, it can be described reasonably simply in the setting of this paper. For a given n, and a collection of points Φ_n, consider the collections of Čech (or Vietoris-Rips) complexes $C(\Phi_n, r)$ built over these points, as r grows. Initially, $C(\Phi_n, 0)$ will contain only the points of Φ_n. However, as r increases, different homological entities (cycles of differing degree) will appear and, eventually, disappear. If to each such phenomenon we assign an interval starting at the birth time and ending at the death time, then the collection of all of these intervals is a representation of the persistent homology generated by Φ_n and is known as its 'barcode'. The individual intervals are referred to as 'bars'.The Betti numbers $\beta_k(C(\Phi_n, r))$ therefore count the number of bars related to k-cycles active at 'connection distance' r.

Heuristics and (unpublished) simulations indicate that bars of regular or negatively correlated point processes start later and vanish earlier than those for Poisson point process. Some of our results confirm this heuristic. For example, using the results above it is easy to see that non-trivial homology groups of Čech and Vietoris-Rips complexes start to appear once r_n satisfies $r_n^{d(k+1)} f^{k+2}(r_n) = \omega(n^{-1})$. For the Poisson case this requires only $r_n = \omega(n^{-1/d(k+1)})$. Since, typically, $f(r) \to 0$ as $r \to 0$, we therefore generally need larger radii for non-trivial homology to appear. The disappearance of homology is harder, however, and in general our results on connectivity cannot confirm the heuristic. However, for the Ginibre point process and zeroes of GEF in $\mathbb{R}^2$, they do show that non-trivial topology vanishes at $r_n = \omega((\log n)^{1/4})$ as opposed to $\omega((\log n)^{1/2})$ for a two dimensional Poisson process.

As for implications to TDA, applied topologists are beginning to appreciate the fact that stochasticity underlies their data as a consequence of sampling, and are beginning to build statistical models to allow parameter estimation and inference (e.g. [8, 11, 26, 34]). The results of this paper show that small changes in model structure (such as the introduction of attraction and repulsion between points in a data cloud) can have measurable effects on topological behaviour.

The remainder of the paper is organised as follows: In the following sec-

tion, we shall summarise some facts needed from the theory of point processes. Sections 3, 4 and 5 contain the core technical results on component and subgraph counts, Betti numbers and Morse critical points respectively. We shall give careful proofs for all the results of Section 3. The results of the Sections 4 and 5 are either easy corollaries of earlier results or can be proved by using similar techniques, and so there we shall give less detail. Section 6 contains a technical result regarding Palm void probabilities of Ginibre process which Manjunath Krishnapur proved for us.

2. Point processes. Our aim in this section is to set up some general definitions related to point processes, give some background on those of main interest to us, and to prove two technical results, of some independent interest, which we shall need later.

2.1. *Point processes and Palm measures.* A point process Φ in $\mathbb{R}^d$ is a $\mathcal{N}$-valued random variable, where $\mathcal{N}$ is the space of locally finite (Radon) counting measures in $\mathbb{R}^d$ equipped with the canonical σ-algebra (cf. [20, 33]). We can represent Φ as either a random measure, $\Phi(\cdot) = \sum_i \delta_{X_i}(\cdot)$, or as a random point set $\Phi = \{X_i\}_{i \geq 1}$, where, in both cases, the X_i are the 'points' of the process.

The factorial moment measure $\alpha^{(k)}$ of a point process Φ is defined by

$$\alpha^{(k)}\left(\prod_{i=1}^n B_i\right) = \mathbb{E}\left\{\prod_{i=1}^n \Phi(B_i)\right\},$$

for disjoint bounded Borel subsets $B_1, \ldots, B_n$. When $k = 1$, $\alpha := \alpha^{(1)}$ is called the *intensity* or *mean measure*, and $\alpha^{(k)}$ also serves as the intensity measure of the point process

$$\Phi^{(k)} := \{(X_1, \ldots, X_k) \in \Phi^k : X_i \neq X_j, \ \forall \ i \neq j\}.$$

The *k-th joint intensity*, $\rho^{(k)} : (\mathbb{R}^d)^k \to [0, \infty)$ is the density (if it exists) of α^k with respect to (in this paper) Lebesgue measure. The $\rho^{(k)}$ characterize the point process just as moments characterize a random variable. Throughout, we shall restrict ourselves to simple stationary point processes of unit intensity; viz. $\alpha(B) = |B|$ for all bounded, Borel B. Most of our results, however, will depend mainly on the behaviour of ρ^k in the neighbourhood of the origin.

For a point process Φ whose probability distribution is $\mathbb{P}$, its reduced *Palm probability distribution* $\mathbb{P}_x^!$ at $x \in \mathbb{R}^d$ is defined as the measure whose corresponding Palm expectation $\mathbb{E}_x^!$ satisfies the following *Campbell-Mecke*

formula (cf. [33, p119]) for any bounded measurable function $h : \mathcal{N} \times \mathbb{R}^d \to \mathbb{R}_+$:

$$(2.3) \qquad \mathbb{E}\left\{\sum_{X \in \Phi} h(\Phi \setminus \{X\}, X)\right\} \; = \; \int_{x \in \mathbb{R}^d} \mathbb{E}_x^!\{h(\Phi, x)\}\, dx.$$

In particular, we assume that $\mathbb{P}_x^!\{\Phi(x) = 0\} = 1$. Intuitively, $\mathbb{P}_x^!$ is the distribution of the remainder of the point process, conditioned on there having been a point at x.

2.2. *Some special cases.* We shall assume the reader is familiar with stationary Poisson point processes, determined, for example, by $\rho^k \equiv 1$ for all k, and use this as a basis for comparison in a quick tour through some non-Poisson cases that will provide examples for the theorems of the remaining sections.

Associated point processes. A point process Φ is called *associated (or positively associated)* if for any finite collection of disjoint bounded Borel subsets $B_1, \ldots, B_k \subset \mathbb{R}^d$ and f, g continuous and increasing functions taking values in $[0, 1]$,

$$(2.4) \quad \mathsf{Cov}\,(f(\Phi(B_1), \ldots, \Phi(B_k)),\; g(\Phi(B_1), \ldots, \Phi(B_k))) \; \geq \; 0,$$

cf. [9]. Examples of associated processes abound. We call a point process Φ *negatively associated* if

$$(2.5) \quad \mathsf{Cov}\,(f(\Phi(B_1), \ldots, \Phi(B_k)),\; g(\Phi(B_{k+1}), \ldots, \Phi(B_l))) \; \leq \; 0,$$

for any finite collection of bounded Borel subsets $B_1, \ldots, B_l \subset \mathbb{R}^d$ such that $(B_1 \cup \cdots \cup B_k) \cap (B_{k+1} \cup \cdots \cup B_l) = \emptyset$ and f, g increasing bounded continuous functions.

As we shall see, examples of negatively associated processes are scarcer than their positively associated counterparts, a phenomenon that occurs even in simpler situations; cf. [28]. The stationary Poisson point process is both negatively and positively associated. Finite independent unions of negatively associated point processes is negatively associated as well and this can be used to construct many examples of negatively associated point processes from a few simple examples. Here are three others that will interest us:

Determinantal and permanental processes.   A simple point process Φ on $\mathbb{R}^d$ is said to be determinantal with kernel $K : (\mathbb{R}^d)^2 \to \mathbb{C}$ if its joint intensities satisfy the following equality for all $k \geq 1$ and for all $x_1, \ldots, x_k \in \mathbb{R}^d$:

(2.6) $$\rho^k(x_1, \ldots, x_k) = \det(K(x_i, x_j)_{1 \leq i,j \leq k}),$$

where det indicates a determinant of a matrix.

Stationary determinantal point processes are negatively associated ([23]). For examples of stationary determinantal point processes, see [21, Section 5]. A determinantal process of particular interest is the unit intensity *Ginibre process* ([2, Sec 4.3.7]) which has kernel

$$K(z, w) = \exp(-\frac{1}{2}(\|z\|^2 + \|w\|^2) + z\overline{w}), \qquad z, w \in \mathbb{C}.$$

For permanental point processes, the determinant in (2.6) is replaced by a matrix permanent.

Perturbed lattices.   Let $N_z : z \in \mathbb{Z}^d$ be iid integer valued random variables distributed as N, and X_{iz}, $i \geq 1$, $z \in \mathbb{Z}^d$ be iid $\mathbb{R}^d$ valued random variables distributed as X. A perturbed lattice is defined as

$$\Phi(N, X) \; := \; \bigcup_{z \in \mathbb{Z}^d} \; \bigcup_{i=1}^{N_z} \{z + X_{iz}\}$$

provided that $\Phi(N, X)$ is a simple point process. N is called the replication kernel and X is called the perturbation kernel. The point process for which $N \equiv 1$ and X is uniform on the unit cube is known as the *simple perturbed lattice* and is negatively associated. For more details see below and especially [6].

Zeroes of a Gaussian entire function.   (Normalized) Gaussian entire functions are defined on the complex plane $\mathbb{C}$ via the a.s. convergent expansion $f(z) = \sum_{n=0}^{\infty} \xi_n z^n / \sqrt{n!}$, where the ξ_n are iid standard complex Gaussians. The zeros of f (when considered as a point process in $\mathbb{R}^2$ and called as zeros of GEF) while neither negatively associated nor determinantal, share many properties with the Ginibre process that make them interesting and tractable; cf. [2] for more background.

Sub- and Super-Poisson processes.   At times, weaker notions than association, based only on factorial moment measures, suffice to establish interesting results.

We say that a point process Φ_1 is α-*weaker* than Φ_2 (written $\Phi_1 \leq_{\alpha-w} \Phi_2$) if $\alpha_1^{(k)}(B) \leq \alpha_2^{(k)}(B)$ for all $k \geq 1$ and bounded Borel $B \subset (\mathbb{R}^d)^k$. We call a point process α-*negatively associated (associated)* if $\alpha^{(k+l)}(B_1 \times B_2) \leq (\geq)\alpha^{(k)}(B_1)\alpha^{(l)}(B_2)$ for all $k, l \geq 1$ and bounded Borel $B_1 \times B_2 \subset (\mathbb{R}^d)^k \times (\mathbb{R}^d)^l$.

Negative association (association) implies α-negative association (association) which in turn implies α-weaker ordering with respect to the Poisson process with intensity measure α.

Even weaker notions of association come from looking at void probabilities, and we say that a point process Φ_1 is ν-*weaker* than Φ_2 (denoted by $\Phi_1 \leq_{\nu-w} \Phi_2$) if

$$\nu_1(B) = \mathbb{P}\left\{\Phi_1(B) = \emptyset\right\} \ \leq \ \mathbb{P}\left\{\Phi_2(B) = \emptyset\right\} = \nu_2(B)$$

for all B bounded Borel subsets.

Finally, we call a point process α-*sub-Poisson (super-Poisson)* if it is α-weaker (stronger) than the Poisson point process and similarly for ν-*sub-Poisson (super-Poisson)*. A point process is *weakly sub-Poisson (super-Poisson)* if it is both α- and ν-sub-Poisson (super-Poisson).

It is known that negative association (association) implies the weak sub-Poisson (super-Poisson) property. Other examples come from perturbed lattices. For example, if the replication kernel N is hypergeometric or binomial, then the resulting point process is a weakly sub-Poisson point process. On the other hand, negative binomial and geometric perturbation kernels lead to weakly super-Poisson processes. Permanental point processes are also weakly sub-Poisson. See [6] for proofs and more about stochastic ordering of point processes.

2.3. *Two technical lemmas.* We shall state some general results about Palm measures of these point processes that we need later. The first lemma shows that negatively associated point processes are "stochastically stronger" than their Palm versions. This can also be viewed as a justification for the usage of negative association as the defining property of sparse point processes. The second shows that Palm versions of negatively associated point processes also exhibit negative association. We state the results in more generality than we need, since they seem to be of independent interest.

LEMMA 2.1. *Let Φ be a negatively associated stationary point process in $\mathbb{R}^d$ of unit intensity and $F : \mathbb{R}^{dn} \to \mathbb{R}_+$ an increasing bounded continuous function. Then for $B_1, \ldots, B_n$ disjoint bounded Borel subsets and almost every $\mathbf{x} \in \mathbb{R}^{dk}$,*

$$(2.7) \quad \mathbb{E}^!_{x_1,\ldots,x_k}\left(F(\Phi(B_1), \ldots, \Phi(B_n))\right) \ \leq \ \mathbb{E}\{F(\Phi(B_1), \ldots, \Phi(B_n))\}.$$

The above inequality will be reversed for an associated point process.

PROOF. For $0 < \epsilon < r$ we have

$$\mathbb{E}\big\{F(\Phi(B_1 \setminus B_\mathbf{x}(r)), \ldots, \Phi(B_n \setminus B_\mathbf{x}(r))) \,\big|\, \Phi(B_{x_i}(\epsilon)) \geq 1, 1 \leq i \leq k\big\}$$

$$= \frac{\mathbb{E}\big\{F(\Phi(B_1 \setminus B_\mathbf{x}(r)), \ldots, \Phi(B_n \setminus B_\mathbf{x}(r))) \prod_{i=1}^k \mathbb{1}[\Phi(B_{x_i}(\epsilon)) \geq 1]\big\}}{\mathbb{P}\{\Phi(B_{x_i}(\epsilon)) \geq 1, 1 \leq i \leq k\}}$$

$$\leq \frac{\mathbb{E}\{F(\Phi(B_1 \setminus B_\mathbf{x}(r)), \ldots, \Phi(B_n \setminus B_\mathbf{x}(r)))\} \mathbb{E}\big\{\prod_{i=1}^k \mathbb{1}[\Phi(B_{x_i}(\epsilon)) \geq 1]\big\}}{\mathbb{P}\{\Phi(B_{x_i}(\epsilon)) \geq 1, 1 \leq i \leq k\}}$$

$$= \mathbb{E}\{F(\Phi(B_1 \setminus B_\mathbf{x}(r)), \ldots, \Phi(B_n \setminus B_\mathbf{x}(r)))\},$$

where the inequality is due to the negative association property of Φ.

Sending first $\epsilon \to 0$ and then $r \to 0$, (2.7) follows immediately from [32, Lemma 6.3] and monotone convergence. $\qquad\square$

LEMMA 2.2. *Let Φ be a negatively associated stationary point process in $\mathbb{R}^d$ of unit intensity. Let $F\!:\!\mathbb{R}^{dn} \to \mathbb{R}_+$ and $G\!:\!\mathbb{R}^{dm} \to \mathbb{R}_+$ be increasing bounded continuous functions. Then for $B_1, \ldots, B_{m+n}$ disjoint bounded Borel subsets and almost every $\mathbf{x} \in \mathbb{R}^{d(k+l)}$,*

$$\mathbb{E}^!_\mathbf{x}\{F(\Phi(B_1), \ldots, \Phi(B_n))\, G(\Phi(B_{n+1}), \ldots, \Phi(B_{m+n}))\}\, \rho^{(k+l)}(\mathbf{x})$$
$$\leq \mathbb{E}^!_{x_1,\ldots,x_k}\{F(\Phi(B_1), \ldots, \Phi(B_n))\}$$
$$\times \mathbb{E}^!_{x_{k+1},\ldots,x_{k+l}}\{G(\Phi(B_{n+1}), \ldots, \Phi(B_{m+n}))\}$$
$$\times \rho^{(k)}(x_1, \ldots, x_k)\rho^{(l)}(x_{k+1}, \ldots, x_{k+l}).$$

The above inequality will be reversed for an associated point process Φ.

PROOF. As in the proof of Lemma 2.1, take $0 < \epsilon < r$. For notational simplicity, set $B^* = B \setminus B_\mathbf{x}(r)$ for bounded Borel set B.

$$\mathbb{E}\Big\{F(\Phi(B_1^*), \ldots, \Phi(B_n^*))\, G(\Phi(B_{n+1}^*), \ldots, \Phi(B_{m+n}^*))$$
$$\big|\, \Phi(B_{x_i}(\epsilon)) \geq 1,\ 1 \leq i \leq (k+l)\Big\}$$
$$\times \mathbb{P}\Big\{\Phi(B_{x_i}(\epsilon)) \geq 1,\ 1 \leq i \leq (k+l)\Big\}$$
$$= \mathbb{E}\Big\{F(\Phi(B_1^*), \ldots, \Phi(B_n^*)) \prod_{i=1}^k \mathbb{1}[\Phi(B_{x_i}(\epsilon)) \geq 1]$$

$$\times G(\Phi(B^*_{n+1}), \ldots, \Phi(B^*_{m+n})) \prod_{i=1}^{l} \mathbb{1}\left[\Phi(B_{x_{k+i}}(\epsilon)) \geq 1\right]\Big\}$$

$$\leq \quad \frac{\mathbb{E}\Big\{F(\Phi(B^*_1), \ldots, \Phi(B^*_n)) \prod_{i=1}^{k} \mathbb{1}\left[\Phi(B_{x_i}(\epsilon)) \geq 1\right]\Big\}}{\mathbb{P}\left\{\Phi(B_{x_i}(\epsilon)) \geq 1, 1 \leq i \leq k\right\}}$$

$$\times \frac{\mathbb{E}\Big\{G(\Phi(B^*_{n+1}), \ldots, \Phi(B^*_{m+n})) \prod_{i=1}^{l} \mathbb{1}\left[\Phi(B_{x_{k+i}}(\epsilon)) \geq 1\right]\Big\}}{\mathbb{P}\left\{\Phi(B_{x_{k+i}}(\epsilon)) \geq 1, 1 \leq i \leq l\right\}}$$

$$\times \mathbb{P}\left\{\Phi(B_{x_i}(\epsilon)) \geq 1, 1 \leq i \leq k\right\} \mathbb{P}\left\{\Phi(B_{x_{k+i}}(\epsilon)) \geq 1, 1 \leq i \leq l\right\}$$

$$= \quad \mathbb{E}\{F(\Phi(B^*_1), \ldots, \Phi(B^*_n)) \mid \Phi(B_{x_i}(\epsilon)) \geq 1, 1 \leq i \leq k\}$$

$$\times \mathbb{E}\{G(\Phi(B^*_{n+1}), \ldots, \Phi(B^*_{m+n})) \mid \Phi(B_{x_{k+i}}(\epsilon)) \geq 1, 1 \leq i \leq l\}$$

$$\times \mathbb{P}\left\{\Phi(B_{x_i}(\epsilon)) \geq 1, 1 \leq i \leq k\right\} \mathbb{P}\left\{\Phi(B_{x_{k+i}}(\epsilon)) \geq 1, 1 \leq i \leq l\right\},$$

where the inequality is due to the negative association of Φ. As in the previous proof, the conditional expectations in the first and last expressions converge to the respective Palm expectations as $\epsilon \to 0$. Since Φ is a simple point process, after dividing by $|B_0(\epsilon)|^{k+l}$ on both sides, the product of the probability terms in the last line converges to $\rho^{(k)}(x_1, \ldots, x_k)\rho^{(l)}(x_{k+1}, \ldots, x_{k+l})$ while the probability term in the first line converges to $\rho^{(k+l)}(\mathbf{x})$ as $\epsilon \to 0$. Complete the proof by sending $r \to 0$. □

3. Subgraph and component counts in random geometric graphs.
Recall that for a point set Φ and radius $r > 0$, the geometric graph $G(\Phi, r)$ is defined as the graph with vertex set Φ and edge-set $\{(X, Y): \|X - Y\| \leq r\}$. We shall work with restrictions of Φ to a sequence of increasing windows $W_n = [\frac{-n^{1/d}}{2}, \frac{n^{1/d}}{2}]^d$, along with a radius regime $\{r_n > 0\}_{n \geq 1}$, setting $\Phi_n := \Phi \cap W_n$. The choice of the radius regime will impact on the asymptotic properties of the geometric graph when the points of Φ are those of a point process.

Let Γ be a connected graph on k vertices. In this section we shall be interested in how often Γ appears (up to graph isomorphisms) in a sequence of geometric graphs $G_n = G(\Phi_n, r_n)$, and how often among such appearances it is actually isomorphic to a component of G_n, viz. it is a Γ-component of G_n. For graphs built over Poisson and iid processes, we know from [30, Chapters 3,13] that no Γ-components exist when $n(r_n^d)^{k-1} \to 0$ ($|\Gamma| = k$), but that they do appear when $n(r_n^d)^{k-1} \to \infty$. The Γ-components continue to exist even when $r_n^d = o(\log n)$ and vanish when $r_n^d = \omega(\log n)$, which is the threshold for connectivity of the graph.

In this section, we shall show, among other things, that the threshold for

formation of Γ-components for negatively associated processes with $r_n \to 0$ is $n(r_n^d)^{k-1} f^k(r_n) \to \infty$, for functions f^k which typically satisfy $f^k(r) \to 0$ as $r \to 0$, and so is higher than in the Poisson case. These components continue to exist even when $r_n^d \to \beta > 0$. The threshold for the vanishing of components will be treated in the next section.

The reader should try to keep this broader picture in mind as she wades through the various limits of this section.

3.1. *Some notation and a start.*   As above, let Γ be a connected graph on k vertices, $k \geq 1$, and $\{x_1, \ldots, x_k\}$ a collection of k points in $\mathbb{R}^d$. Introduce the (indicator) function $h_\Gamma : \mathbb{R}^{dk} \times \mathbb{R}_+ \to \{0,1\}$ by

$$(3.8) \qquad h_\Gamma(\mathbf{x}, r) \quad := \quad \mathbb{1}[G(\{x_1, \ldots, x_k\}, r) \simeq \Gamma],$$

where $\simeq$ denotes graph isomorphism and $\mathbb{1}$ is the usual indicator function. For a fixed sequence $\{r_n\}$ set

$$(3.9) \qquad h_{\Gamma,n}(\mathbf{x}) \quad := \quad h_\Gamma(\mathbf{x}, r_n),$$

and, for $r = 1$, write

$$(3.10) \qquad h_\Gamma(\mathbf{x}) \quad := \quad h_\Gamma(\mathbf{x}, 1)$$

Moving now to the random setting, in which Φ is a simple point process with k-th intensities $\rho^{(k)}$, we say that Γ is *a feasible subgraph* of Φ if

$$\int_{(\mathbb{R}^d)^k} h_\Gamma(\mathbf{x}) \rho^{(k)}(\mathbf{x}) \, d\mathbf{x} \quad > \quad 0.$$

Thus, Γ is a feasible subgraph of Φ if the $\alpha^{(k)}$ measure of finding a copy of it (up to graph homomorphism) in $G(\Phi, 1)$ is positive. In most cases, feasibility will hold because $\rho^{(k)}(\mathbf{x}) > 0$ a.e. or at least on a large enough set.

We shall be interested in the the number of Γ-subgraphs, $G_n(\Phi, \Gamma)$, and number of Γ-components, $J_n(\Phi, \Gamma)$, of Φ_n, which are defined as follows[3]:

$$(3.11) \qquad G_n(\Phi, \Gamma) \quad := \quad \frac{1}{k!} \sum_{X \in \Phi_n^{(k)}} h_{\Gamma,n}(X)$$

---

[3]In the terminology of [30], $G_n(\Phi, \Gamma)$ denotes the number of induced $\Gamma-$ subgraphs of $G(\Phi, r)$ and not the number of subgraphs of $G(\Phi, r)$ isomorphic to Γ. However, it is easy to see that the latter is a finite linear combination of the number of induced subgraphs of the same order.

$$J_n(\Phi, \Gamma) \quad := \quad \frac{1}{k!} \sum_{X \in \Phi_n^{(k)}} h_{\Gamma,n}(X) \mathbb{1}[\Phi_n(B_X(r_n)) = k].$$

Note that J_n considers graphs based on vertices in Φ_n only, viz. all vertices that lie in W_n. Such a graph, however, may have vertices within distance r_n of a point in the complement of W_n, and so actually be part of something larger. To account for this boundary effect, we introduce an additional variable, which does not count such "boundary crossing" graphs. This is given by

$$(3.12) \qquad \tilde{J}_n(\Phi, \Gamma) := \frac{1}{k!} \sum_{X \in \Phi_n^{(k)}} h_{\Gamma,n}(X) \mathbb{1}[\Phi(B_X(r_n)) = k].$$

We shall see later that in the sparse and thermodynamic regimes the differences between J_n and $\tilde{J}_n$ disappear in asymptotic results. Nevertheless, both are needed for the proofs.

The key ingredient in obtaining asymptotics for sub-graph counts and component counts are the following closed-form expressions, which are immediate consequences of the Campbell-Mecke formula.

$$(3.13) \quad \mathbb{E}\{G_n(\Phi, \Gamma)\} \quad = \quad \frac{1}{k!} \int_{W_n^k} h_{\Gamma,n}(\mathbf{x})\, \rho^{(k)}(\mathbf{x})\, d\mathbf{x},$$

$$(3.14) \quad \mathbb{E}\{J_n(\Phi, \Gamma)\} \quad = \quad \frac{1}{k!} \int_{W_n^k} h_{\Gamma,n}(\mathbf{x})\, \mathbb{P}_{\mathbf{x}}^{!}\{\Phi_n(B_{\mathbf{x}}(r_n)) = 0\}\, \rho^{(k)}(\mathbf{x})\, d\mathbf{x}.$$

Much of the remainder of this section is based on obtaining asymptotic expressions for these integrals in terms of basic point process parameters in the sparse and thermodynamic regimes, as well as looking at bounds on variances. We shall consider the connectivity regime only in the following section on Betti numbers. Our results here extend those of [30, Chapter 3] for Poisson and iid processes, and the general approach of the proofs is thus similar.

3.2. *Sparse regime: $r_n \to 0$.* The intuition behind the following theorem is that in the sparse regime it is difficult to find Γ-subgraphs in a random geometric graph, and even more unlikely that any such subgraph will have another point of the point process near it. This implies that any such sub-graph will actually be a component of the full graph, disconnected from other components.

THEOREM 3.1. *Let Φ be a stationary point process in $\mathbb{R}^d$ of unit intensity and Γ be a feasible connected graph of Φ on k vertices. Let $\rho^{(k)}$ be almost*

everywhere continuous. Assume that $\rho^{(k)}(0,\ldots,0) = 0$, and that there exist functions $f_\rho^k{:}\mathbb{R}_+ \to \mathbb{R}_+$ and $g_\rho^k{:}(B_0(k))^k \to \mathbb{R}_+$ such that

$$\rho^{(k)}(r\mathbf{y}) = \Theta(f_\rho^k(r)) \quad and \quad \lim_{r \to 0} \frac{\rho^{(k)}(r\mathbf{y})}{f_\rho^k(r)} = g_\rho^k(\mathbf{y}),$$

for all $\mathbf{y}$ of the form $\mathbf{y} = (0, y_2, \ldots, y_k)$. Further, assume that $f^{k+1} = O(f^k)$ as $r \to 0$ and g_ρ^k is almost everywhere continuous. Let $r_n \to 0$. Then,

$$(3.15) \quad \lim_{n \to \infty} \frac{\mathbb{E}\{G_n(\Phi,\Gamma)\}}{nr_n^{d(k-1)}f^k(r_n)} = \lim_{n \to \infty} \frac{\mathbb{E}\{J_n(\Phi,\Gamma)\}}{nr_n^{d(k-1)}f^k(r_n)}$$
$$= \mu_0(\Phi,\Gamma)$$
$$:= \begin{cases} 1 & k = 1, \\ \frac{1}{k!}\int_{\mathbb{R}^{d(k-1)}} h_\Gamma(\mathbf{y})g_\rho^k(\mathbf{y})\,d\mathbf{y} & k \geq 1. \end{cases}$$

If $\rho^{(k)}(0,\ldots,0) > 0$, then the same result holds with $f_\rho^k \equiv 1$ and $g_\rho^k \equiv \rho^{(k)}(0,\ldots,0)$.

Before turning to the proof of the theorem we shall make a few points about its conditions, and provide some examples. Throughout we assume that all point processes are normalized to have unit intensity.

1: Note that the theorem does not guarantee the positivity of $\mu_0(\Phi,\Gamma)$.

2: $f^1(r) \equiv 1$ for all stationary point processes of unit intensity since, in this case, $\rho^{(1)} \equiv 1$.

3: It is easy to check that if Φ is α-negatively associated or α-super-Poisson, then the condition $f^{k+1} = O(f^k)$ as $r \to 0$ is satisfied.

4: In the case $\rho^k(0,\ldots,0) = 0$ for $k \geq 2$, even if we cannot find appropriate f^k or g_ρ^k, it is still true that $\mathbb{E}\{G_n(\Phi,\Gamma)\} = o(nr_n^{d(k-1)})$.

5: If Φ is only $\mathbb{Z}^d$-stationary (as is the case with perturbed lattices), then it will be clear from the proof that (3.16) still holds, but with

$$\mu_0(\Phi,\Gamma) := \frac{1}{k!}\int_{[0,1]^d}\int_{\mathbb{R}^{d(k-1)}} h_\Gamma(x,\mathbf{y})g_\rho^k(x,\mathbf{y})\,dx\,d\mathbf{y}.$$

6: For a homogeneous Poisson point process, the theorem holds with $f^k \equiv 1$ and $g_\rho^k \equiv 1$, recovering [30, Prop. 3.1].

7: If $\Phi \geq_{\alpha-w} \Phi_{(1)}$, then for all $k \geq 1$, $\rho^{(k)} \geq \rho_{(1)}^{(k)} \equiv 1$ and hence $f^k \equiv 1$ and also $\mu_0(\Phi,\Gamma) > 0$. Examples of point processes in this class are all super-Poisson perturbed lattices and permanental point processes.

8: For a perturbed lattice Φ with perturbation kernel $N \in \{0, \ldots, K\}$ a.s., $\rho^{(k)}(0, \ldots, 0) > 0$ iff $k \leq K$. In this case, $\mu_0(\Phi, \Gamma) > 0$ for a connected graph Γ on k vertices. For connected graphs Γ on k vertices with $k > K$, $nr_n^{-d(k-1)}\mathbb{E}\{G_n(\Phi, \Gamma)\} \to 0$. For sub-Poisson perturbed lattices, the existence of f^k depends on the perturbation kernel. However, for high values of k, it is clear that the scaling for sub-Poisson perturbed lattices will differ significantly from that of the Poisson case.

9: From [27, Theorem 1.1] for the zeroes of Gaussian entire function and calculations similar to [2, Theorem 4.3.10] for the Ginibre point process, one can check that in both cases

$$\rho^k(x_1, \ldots, x_k) = \Theta\Big(\prod_{i<j} \|x_i - x_j\|^2\Big).$$

Hence, $f^k(r) = \Theta(r^{k(k-1)})$ for these processes.

PROOF. We shall prove the theorem for $k \geq 2$. The case $k = 1$ follows easily by making a few notational changes to the general case. We start with the convergence of $\mathbb{E}\{G_n(\Phi, \Gamma)\}$. In the expression for $\mathbb{E}\{G_n(\Phi, \Gamma)\}$ in (3.13), make the change of variable $x_i = x_1 + r_n y_i$ for $i \geq 2$ and then use stationarity of the point process to obtain

$$
\begin{aligned}
\mathbb{E}\{G_n(\Phi, \Gamma)\} &= \frac{r_n^{d(k-1)}}{k!} \int_{W_n} \int_{(r_n^{-1}(W_n - x))^{k-1}} h_{\Gamma,n}(x\mathbf{1} + r_n\mathbf{y}) \\
&\qquad\qquad\qquad\qquad\qquad\times \rho^{(k)}(x\mathbf{1} + r_n\mathbf{y}) \, dx \ldots dy_k \\
(3.16) \qquad &= \frac{r_n^{d(k-1)}}{k!} \int_{W_n} \int_{(r_n^{-1}(W_n - x))^{k-1}} h_{\Gamma,n}(r_n\mathbf{y})\rho^{(k)}(r_n\mathbf{y}) \, dx \ldots dy_k \\
&\leq \frac{r_n^{d(k-1)}}{k!} \int_{W_n} \int_{\mathbb{R}^{d(k-1)}} h_{\Gamma}(\mathbf{y})\rho^{(k)}(r_n\mathbf{y}) \, dx \ldots dy_k \\
&= \frac{nr_n^{d(k-1)}}{k!} \int_{\mathbb{R}^{d(k-1)}} h_{\Gamma}(\mathbf{y})\rho^{(k)}(r_n\mathbf{y}) \, d\mathbf{y}.
\end{aligned}
$$

Since Γ is a connected graph, $h_{\Gamma} \equiv 0$ outside $(B_0(k))^{k-1}$ and hence the preceding integral is finite. Further for all $x \in W_{(n^{\frac{1}{d}} - 2k)^d}$, it follows that

$$B_O(k) \subset (W_n - x) \subset r_n^{-1}(W_n - x)$$

for large n. Hence, for large enough n,

$$\mathbb{E}\{G_n(\Phi, \Gamma)\} \geq \frac{r_n^{d(k-1)}}{k!} \int_{W_{(n^{\frac{1}{d}} - 2k)^d}} \int_{(r_n^{-1}(W_n - x))^{k-1}} h_{\Gamma}(\mathbf{y})\rho^{(k)}(r_n\mathbf{y}) \, dx \ldots dy_k$$

$$= \frac{r_n^{d(k-1)}}{k!} \int_{W_{(n^{\frac{1}{d}}-2k)^d}} \int_{\mathbb{R}^{d(k-1)}} h_\Gamma(\mathbf{y}) \rho^{(k)}(r_n\mathbf{y}) \, dx \ldots dy_k$$

$$= \frac{((n^{\frac{1}{d}}-2k)^d) r_n^{d(k-1)}}{k!} \int_{\mathbb{R}^{d(k-1)}} h_\Gamma(\mathbf{y}) \rho^{(k)}(r_n\mathbf{y}) \, d\mathbf{y}.$$

Thus, as $n \to \infty$,

$$\frac{\mathbb{E}\{G_n(\Phi,\Gamma)\}}{nr_n^{d(k-1)}} \sim \frac{1}{k!} \int_{\mathbb{R}^{d(k-1)}} h_\Gamma(\mathbf{y}) \rho^{(k)}(r_n\mathbf{y}) \, d\mathbf{y}.$$

Note that we can restrict the range of integration in the above equation to $B_0(k)$. Since $\rho^{(k)}(r_n\mathbf{y})/f^k(r_n) = g_\rho^k(\mathbf{y})$ a.e. in $B_0(k)$ and g_ρ^k is bounded (as it is continuous) in $B_0(k)$, we can use the Lebesgue dominated convergence theorem to show that, as $n \to \infty$,

$$\frac{\mathbb{E}\{G_n(\Phi,\Gamma)\}}{nr_n^{d(k-1)}f^k(r_n)} \to \mu_0(\Phi,\Gamma).$$

This proves the convergence of expected number of Γ-subgraphs.

We shall now show that the normalised expected numbers of components and subgraphs are asymptotically equivalent for small enough radii. This will complete the proof of the theorem.

Using the lower bound of $1 - \Phi(B_X(r_n))$ for the void term in J_n (see (3.11)) we obtain the following lower bound for J_n.

$$J_n(\Phi,\Gamma) \geq G_n(\Phi,\Gamma) - \frac{1}{k!} \sum_{X \in \Phi_n^{(k)}} h_{\Gamma,n}(X)\Phi(B_X(r_n)) = G_n(\Phi,\Gamma) - E_n(\Phi,\Gamma).$$

Since $J_n \leq G_n$, we only need to show that $\frac{\mathbb{E}\{E_n(\Phi,\Gamma)\}}{nr_n^{d(k-1)}f^k(r_n)} \to 0$. From the Campbell-Mecke formula, we have

$$\mathbb{E}\{E_n(\Phi,\Gamma)\} = \frac{1}{k!} \int_{W_n^k} h_{\Gamma,n}(\mathbf{x})\mathbb{E}_\mathbf{x}^!\{\Phi(B_\mathbf{x}(r_n))\}\rho^{(k)}(\mathbf{x}) \, d\mathbf{x}.$$

From [32, Lemma 6.4], we know that $\rho_\mathbf{x}^{!(1)}(y) = \frac{\rho^{(k+1)}(\mathbf{x},y)}{\rho^{(k)}(\mathbf{x})}$. Now applying the Campbell-Mecke formula for $\mathbb{E}_\mathbf{x}^!$ in the above equation, we find that

$$\mathbb{E}\{E_n(\Phi,\Gamma)\} = \frac{1}{k!} \int_{W_n^k \times B_\mathbf{x}(r_n)} h_{\Gamma,n}(\mathbf{x})\rho^{(k+1)}(\mathbf{x},y) \, d\mathbf{x} \, dy.$$

Now apply the change of variables $x_i = x_1 + r_n y_i$ for $i \geq 2$, $y = r_n y$ and proceed as in the case of $\mathbb{E}\{G_n\}$, to see that, for large enough n,

$$\mathbb{E}\{E_n(\Phi, \Gamma)\} \leq \frac{nr_n^{dk}}{k!} \int_{\mathbb{R}^{d(k-1)} \times B_0(k+1)} h_\Gamma(\mathbf{y}) \rho^{(k+1)}(r_n \mathbf{y}, r_n y) \, d\mathbf{y} \, dy,$$

where the additional factor of r_n^d is due to the y variable. Dividing by $nr_n^{d(k-1)} f^k(r_n)$ and bounding h_Γ by 1, we have

$$\frac{\mathbb{E}\{E_n(\Phi, \Gamma)\}}{nr_n^{d(k-1)} f^k(r_n)} \leq \frac{r_n^d f^{k+1}(r_n)}{k! f^k(r_n)} \int_{B_0(k)^{k-1} \times B_0(k+1)} \frac{\rho^{(k+1)}(r_n \mathbf{y}, r_n y)}{f^{k+1}(r_n)} \, dy \, d\mathbf{y}.$$

Since $f^{k+1}(r) = O(f^k(r))$ by assumption, $\frac{\mathbb{E}\{E_n(\Phi, \Gamma)\}}{nr_n^{d(k-1)} f^k(r_n)} \to 0$ and hence

$$\frac{\mathbb{E}\{J_n(\Phi, \Gamma)\}}{nr_n^{d(k-1)} f^k(r_n)} \to \mu_0(\Phi, \Gamma),$$

as required. $\square$

The following corollary follows easily from the ordering of the joint intensities of the point processes.

COROLLARY 3.2.   *Let Φ_i, $i = 1, 2$, be two stationary point processes and $f_{\rho_i}^k$, $g_{\rho_i}^k$ correspond to the functions of Theorem 3.1. If $\Phi_1 \leq_{\alpha-w} \Phi_2$, then $f_{\rho_1}^k \leq f_{\rho_2}^k$. If $f_{\rho_1}^k \equiv f_{\rho_2}^k$, then $g_{\rho_1}^k \leq g_{\rho_2}^k$, and hence $\mu_0(\Phi_1, \Gamma) \leq \mu_0(\Phi_2, \Gamma)$ for a connected graph Γ that is feasible for both Φ_1 and Φ_2.*

3.3. *Thermodynamic regime: $r_n^d \to \beta$.*

THEOREM 3.3.   *Let Φ be a stationary point process in $\mathbb{R}^d$ of unit intensity and Γ be a feasible connected graph of Φ on k vertices. Assume that $\rho^{(k)}$ is almost everywhere continuous, and let $r_n^d \to \beta > 0$ and $\mathbf{y} = (0, y_2, \ldots, y_k)$. Then,*

$$\lim_{n \to \infty} \frac{\mathbb{E}\{G_n(\Phi, \Gamma)\}}{n} = \mu_\beta(\Phi, \Gamma)$$

$$(3.17) \qquad := \begin{cases} 1 & k = 1, \\ \frac{\beta^{k-1}}{k!} \int_{\mathbb{R}^{d(k-1)}} h_\Gamma(\mathbf{y}) \rho^{(k)}(\beta^{\frac{1}{d}} \mathbf{y}) \, d\mathbf{y} & k \geq 2, \end{cases}$$

$$\lim_{n \to \infty} \frac{\mathbb{E}\{J_n(\Phi, \Gamma)\}}{n} = \gamma_\beta(\Phi, \Gamma)$$

$$(3.18) \qquad := \quad \begin{cases} \mathbb{P}^!_O \left\{ \Phi(B_O(\beta^{\frac{1}{d}})) = 0 \right\} & k = 1, \\ \frac{\beta^{k-1}}{k!} \int_{\mathbb{R}^{d(k-1)}} h_\Gamma(\mathbf{y}) \rho^{(k)}(\beta^{\frac{1}{d}}\mathbf{y}) & \\ \qquad \times \mathbb{P}^!_{\beta^{\frac{1}{d}}\mathbf{y}} \left\{ \Phi(B_{\beta^{\frac{1}{d}}\mathbf{y}}(\beta^{\frac{1}{d}})) = 0 \right\} \, d\mathbf{y} & k \geq 2. \end{cases}$$

If Φ is a negatively associated point process with $\mathbb{P}\left\{ \Phi(B_{\mathbf{x}}(\beta^{\frac{1}{d}})) = 0 \right\} > 0$ for almost every $\mathbf{x} \in B_0(\beta^{\frac{1}{d}}k)^k$, then $\gamma_\beta(\Phi, \Gamma) > 0$.

Again, before turning to the proof, we make some observations about the theorem:

1: The positivity of $\gamma_\beta(\Phi, \Gamma)$ is not immediate. For an example in which this does not hold, let Φ_0 be a Poisson point process of unit intensity in $\mathbb{R}^d$, Φ_i, $i \geq 1$ iid copies of the point process of 4 iid uniformly distributed points in $B_O(\beta^{\frac{1}{d}}/2)$, and define the Cox point process,

$$\Phi \quad := \quad \bigcup_{X_i \in \Phi_0} \{X_i + \Phi_i\}.$$

Clearly, for all $X \in \Phi$, $\mathbb{P}\left\{ \Phi(B_X(\beta^{\frac{1}{d}}) \geq 4 \right\} = 1$.

Now take $r_n^d \equiv \beta$ and Γ a triangle, and note that $J_n(\Phi, \Gamma) = 0$ for all $n \geq 1$ and so $\gamma_\beta(\Phi, \Gamma) = 0$, even though all the assumptions of Theorem 3.3 are satisfied.

2: As in Corollary 3.2, $\Phi_1 \leq_{\alpha-w} \Phi_2$ implies that $\mu_\beta(\Phi_1, \Gamma) \leq \mu_\beta(\Phi_2, \Gamma)$. However, as the previous example shows, the situation for $\gamma_\beta(\Phi, \Gamma)$ is somewhat more complicated.

3: If $|\Gamma| = 1$, then $J_n(\Phi, \Gamma)$ is the number of isolated nodes in the Boolean model of balls of radii β centred on the points of Φ The Palm measure of a determinantal point process is also determinantal and in particular, for the Ginibre process, $\rho^{!(1)}(z) = 1 - e^{-\|z\|^2}$. Using this explicit structure, it can be shown that, for small enough β,

$$\gamma_\beta(\Phi_{Gin}, \Gamma) \geq 1 - \pi\beta^2 + \pi(1 - e^{-\beta^2}) > 1 - \pi\beta^2 + O(\pi^2\beta^4) = \gamma_\beta(\Phi_{Poi}, \Gamma),$$

and hence the inequality for the γ_β could be reversed in the thermodynamic regime for even negatively associated point processes as compared to the sparse regime.

PROOF. Since the proof here is similar to the preceding one, we shall not give all the details and, again, we shall only bother with the case $k \geq 2$. Starting with (3.13) and (3.14), the proof follows similar lines to that

of Theorem 3.1. The difference is that $r_n^{d(k-1)} \to \beta^{k-1}$ and $\rho^{(k)}(r_n\mathbf{y}) \to \rho^{(k)}(\beta^{\frac{1}{d}}\mathbf{y})$ and so there is no need for additional scaling. For the convergence of J_n, one first shows the convergence of $\tilde{J}_n$ using similar techniques to those in the proof of Theorem 3.1. Then, note that

$$\tilde{J}_n(\Phi,\Gamma) \leq J_n(\Phi,\Gamma) \leq \tilde{J}_n(\Phi,\Gamma) + G_n(\Phi_n/\Phi_{(n^{\frac{1}{d}}-(k+1)r_n)^d},\Gamma).$$

The rightmost term in the upper bound accounts for the boundary effects, and by arguments similar to those in the proof of Theorem 3.1, it is easy to see that

$$\mathbb{E}\left\{G_n(\Phi_n/\Phi_{(n^{\frac{1}{d}}-(k+1)r_n)},\Gamma)\right\} = O(|W_n/W_{(n^{\frac{1}{d}}-(k+1)r_n)^d}|) = O(n^{\frac{d-1}{d}}).$$

More importantly for us, this expectation is $o(n)$, and so of lower order than $\tilde{J}_n(\Phi,\Gamma)$. Thus $\mathbb{E}\{J_n(\Phi,\Gamma)\}/n$ also converges to $\gamma_\beta(\Phi,\Gamma)$. Since $r_n^d \to \beta > 0$, the void probability term in J_n is not necessarily degenerate.

The positivity of $\gamma_\beta(\Phi,\Gamma)$ for negatively associated point processes is an easy corollary of Lemma 2.1. We need only note that

$$F(\Phi(B)) = \mathbb{1}[\Phi(B) = 0] = (1 - \Phi(B)) \vee 0$$

is a decreasing bounded continuous function and hence

$$\mathbb{P}_\mathbf{x}^!(\Phi(B_\mathbf{x}(\beta^{\frac{1}{d}})) = 0) \geq \mathbb{P}\left\{\Phi(B_\mathbf{x}(\beta^{\frac{1}{d}})) = 0\right\} > 0$$

for a.e. $\mathbf{x} \in B_0(k)^k$. This completes the proof.   □

3.4. *Variance bounds for the sparse and thermodynamic regimes.* The crux of the second moment bounds lies in the fact that, up to constants, variances are essentially bounded above (below) by expectations for negatively associated (associated) point processes. (It is simple to check that $\mathsf{Var}(.) = \Theta(\mathbb{E}\{.\})$ for the Poisson process, which is both negatively associated and associated (cf. [30, Chapter 3]).) We, however, shall need to extend these inequalities to graph variables, and this is the content of this section.

THEOREM 3.4 (Covariance bounds in sparse regime).   *Let Γ and Γ_0 be two feasible connected graphs on k and l ($k \geq l \geq 2$) vertices, respectively, for a stationary point process Φ with almost everywhere continuous joint densities. Let Φ satisfy the assumptions of Theorem 3.1 and assume that the f^j and g_ρ^j exist for all $j \leq k + l$. Further, let $r_n \to 0$ and $\mu_0(\Phi,\Gamma) > 0$.*

1. *If Φ is α-negatively associated, then*

$$\mathsf{Cov}\,(G_n(\Phi,\Gamma), G_n(\Phi,\Gamma_0)) = O(\mathbb{E}\{G_n(\Phi,\Gamma)\}).$$

2. *If Φ is α-associated, then*

$$\mathsf{Cov}\,(G_n(\Phi,\Gamma), G_n(\Phi,\Gamma_0)) = \Omega(\mathbb{E}\{G_n(\Phi,\Gamma)\}).$$

PROOF. We shall prove the result for α-negatively associated processes and $k \geq 2$. The α-associated case follows by reversing the inequality sign in (3.19) below and the case $k = 1$ needs a few simple notational changes. We shall again use the Campbell-Mecke formula to obtain closed-form expressions for the second moments and then perform a similar analysis as in the proof of Theorem 3.1 to obtain the asymptotics.

For $j \leq l$ and $\mathbf{x} = (x_1, \ldots, x_{k+l-j})$, in analogy to (3.8)–(3.10), define

$$
\begin{aligned}
h_{\Gamma,\Gamma_0,j}(\mathbf{x}) &:= h_\Gamma(x_1, \ldots, x_k) h_{\Gamma_0}(x_1, \ldots, x_j, x_{k+1}, \ldots, x_{k+l-j}), \\
h_{\Gamma,\Gamma_0,j,n}(\mathbf{x}) &:= h_{\Gamma,n}(x_1, \ldots, x_k) h_{\Gamma_0,n}(x_1, \ldots, x_j, x_{k+1}, \ldots, x_{k+l-j}).
\end{aligned}
$$

Then

$$
\begin{aligned}
(3.19) \quad & \mathbb{E}\{G_n(\Phi,\Gamma) G_n(\Phi,\Gamma_0)\} \\
&= \mathbb{E}\left\{ \sum_{\mathcal{X},\mathcal{Y}\subset\Phi, |\mathcal{X}|=k, |\mathcal{Y}|=l} h_{\Gamma,n}(\mathcal{X}) h_{\Gamma_0,n}(\mathcal{Y}) \right\} \\
&= \sum_{j=0}^{l} \mathbb{E}\left\{ \sum_{\mathcal{X},\mathcal{Y}\subset\Phi, |\mathcal{X}|=k, |\mathcal{Y}|=l, |\mathcal{X}\cap\mathcal{Y}|=j} h_{\Gamma,n}(\mathcal{X}) h_{\Gamma_0,n}(\mathcal{Y}) \right\} \\
&= \sum_{j=0}^{l} \frac{1}{j!(k-j)!(l-j)!} \int_{W_n^{k+l-j}} h_{\Gamma,\Gamma_0,j,n}(\mathbf{x}) \rho^{(k+l-j)}(\mathbf{x})\, d\mathbf{x} \\
&\leq \sum_{j=1}^{l} \frac{1}{j!(k-j)!(l-j)!} \int_{W_n^{k+l-j}} h_{\Gamma,\Gamma_0,j,n}(\mathbf{x}) \rho^{(k+l-j)}(\mathbf{x})\, d\mathbf{x} \\
&\qquad + \frac{1}{k!l!} \int_{W_n^k} \int_{W_n^l} h_{\Gamma,\Gamma_0,0,n}(\mathbf{x}) \rho^{(k)}(x_1, \ldots, x_k) \\
&\qquad\qquad \times \rho^{(l)}(x_{k+1}, \ldots, x_{k+l})\, dx_1 \ldots dx_{k+l} \\
&= \sum_{j=1}^{l} \frac{1}{j!(k-j)!(l-j)!} \int_{W_n^{k+l-j}} h_{\Gamma,\Gamma_0,j,n}(\mathbf{x}) \rho^{(k+l-j)}(\mathbf{x})\, d\mathbf{x} \\
&\qquad + \mathbb{E}\{G_n(\Phi,\Gamma)\} \, \mathbb{E}\{G_n(\Phi,\Gamma_0)\},
\end{aligned}
$$

where the inequality is due to the α-negative association property. Thus using similar arguments as in the proof of Theorem 3.1 and setting $\mathbf{y} = (0, y_2 \ldots, y_{k+l-j})$, we have

$$
\begin{aligned}
\mathsf{Cov}\,&(G_n(\Phi,\Gamma), G_n(\Phi,\Gamma_0)) \\
&\leq \sum_{j=1}^{l} \frac{1}{j!(k-j)!(l-j)!} \int_{W_n^{k+l-j}} h_{\Gamma,\Gamma_0,j,n}(\mathbf{x})\rho^{(k+l-j)}(\mathbf{x})\,d\mathbf{x} \\
&\sim \sum_{j=1}^{l} \frac{nr_n^{d(k+l-j-1)}f^{k+l-j}(r_n)}{j!(k-j)!(l-j)!} \int_{\mathbb{R}^{d(k+l-j-1)}} h_{\Gamma,\Gamma_0,j}(\mathbf{y})g_\rho^{(k+l-j)}(\mathbf{y})\,d\mathbf{y} \\
&= O(nr_n^{d(k-1)}f^{k}(r_n)) \\
&= O(\mathbb{E}\{G_n(\Phi,\Gamma)\}),
\end{aligned}
$$

which is what we needed to show. $\square$

Unlike the sparse regime, subgraph counts and component counts have different limits in the thermodynamic regime and hence we need variance bounds on component counts in the thermodynamic regime.

THEOREM 3.5 (Variance bounds in the thermodynamic regime).   *Let Φ be a stationary point process in $\mathbb{R}^d$ of unit intensity and Γ be a feasible connected graph of Φ on k vertices. Assume that $\rho^{(k)}$ is almost everywhere continuous. Let $r_n^d \to \beta > 0$ and $\gamma_\beta(\Phi,\Gamma) > 0$.*

1. *If Φ is negatively associated, then $\mathsf{Var}\left(\tilde{J}_n(\Phi,\Gamma)\right) = O(\mathbb{E}\{J_n(\Phi,\Gamma)\})$.*

2. *If Φ is associated, then $\mathsf{Var}\left(\tilde{J}_n(\Phi,\Gamma)\right) = \Omega(\mathbb{E}\{J_n(\Phi,\Gamma)\})$.*

PROOF. Firstly, write

$$
\tilde{J}_n(\Phi,\Gamma)^2 \;=\; \tilde{J}_n(\Phi,\Gamma) + \sum_{X,Y \subset \Phi_n, |X|=|Y|=k} h_{\Gamma,n}(X)h_{\Gamma,n}(Y) \\
\times \mathbb{1}[\Phi(B_X(r_n)) = \Phi(B_Y(r_n)) = 0].
$$

By the Campbell-Mecke formula,

$$
\begin{aligned}
\mathbb{E}\left\{\tilde{J}_n(\Phi,\Gamma)^2\right\} \;=\; & \mathbb{E}\left\{\tilde{J}_n(\Phi,\Gamma)\right\} + \frac{1}{(k!)^2} \int_{W_n^k \times W_n^k} h_{\Gamma,n}(\mathbf{x})h_{\Gamma,n}(\mathbf{y}) \\
& \times \mathbb{1}[G(\{\mathbf{x},\mathbf{y}\};r_n) \text{ is disconnected}] \\
& \times \mathbb{P}^!_{\mathbf{x},\mathbf{y}}\{\Phi(B_\mathbf{x}(r_n)) = \Phi(B_\mathbf{y}(r_n)) = 0\}\rho^{(2k)}(\mathbf{x},\mathbf{y})\,d\mathbf{x}\,d\mathbf{y}.
\end{aligned}
$$

Thus,

$$\begin{aligned} \mathsf{Var}\Big(\tilde{J}_n(\Phi,\Gamma)\Big) &= \mathbb{E}\Big\{\tilde{J}_n(\Phi,\Gamma)\Big\} \\ &\quad + \frac{1}{(k!)^2}\int_{W_n^k\times W_n^k} h_{\Gamma,n}(\mathbf{x})h_{\Gamma,n}(\mathbf{y})Q_n(\mathbf{x},\mathbf{y})\,d\mathbf{x}\,d\mathbf{y}, \end{aligned}$$

where

$$\begin{aligned} Q_n(\mathbf{x},\mathbf{y}) &:= \mathbb{1}[G(\{\mathbf{x},\mathbf{y}\};r_n) \text{ is disconnected}] \\ &\quad \times \mathbb{P}^!_{\mathbf{x},\mathbf{y}}\{\Phi(B_{\mathbf{x}}(r_n)) = \Phi(B_{\mathbf{y}}(r_n)) = 0\} \\ &\quad \times \rho^{(2k)}(\mathbf{x},\mathbf{y}) - \mathbb{P}^!_{\mathbf{x}}\{\Phi(B_{\mathbf{x}}(r_n)) = 0\} \\ &\quad \times \mathbb{P}^!_{\mathbf{y}}\{\Phi(B_{\mathbf{y}}(r_n)) = 0\}\rho^{(k)}(\mathbf{x})\rho^{(k)}(\mathbf{y}). \end{aligned}$$

Choose n large enough so that $r_n \leq \beta^{\frac{1}{d}} + \frac{1}{4}$. For such an n and negatively associated Φ, we know from Lemma 2.2 that $Q_n(\mathbf{x},\mathbf{y}) \leq 0$ for all $\mathbf{x},\mathbf{y}$ such that the set distance $d_S(\mathbf{x},\mathbf{y}) := \inf_{i,j}\|x_i - y_j\| > 3\beta^{\frac{1}{d}}$. Thus, we have that

$$\begin{aligned} \mathsf{Var}\Big(\tilde{J}_n(\Phi,\Gamma)\Big) &\leq \mathbb{E}\Big\{\tilde{J}_n(\Phi,\Gamma)\Big\} + \frac{1}{(k!)^2}\int_{W_n^k\times W_n^k} h_{\Gamma,n}(\mathbf{x})h_{\Gamma,n}(\mathbf{y}) \\ &\quad \times Q_n(\mathbf{x},\mathbf{y})\mathbb{1}[d(\mathbf{x},\mathbf{y}) \leq 3\beta^{\frac{1}{d}}]\,d\mathbf{x}\,d\mathbf{y}. \end{aligned}$$

From Theorem 3.3, we know that $\mathbb{E}\Big\{\tilde{J}_n(\Phi,\Gamma)\Big\} = \Theta(n)$ and using similar methods as in that theorem, one can show that the latter term in the above equation is of $O(n)$. Combining the two, we get that $\mathsf{Var}\Big(\tilde{J}_n(\Phi,\Gamma)\Big) = O(n)$.

The proof for associated point processes is same as above except that $Q_n(\mathbf{x},\mathbf{y}) \geq 0$ for all $\mathbf{x},\mathbf{y}$ such that $d_S(\mathbf{x},\mathbf{y}) > 3\beta^{\frac{1}{d}}$, giving that in this case $\mathsf{Var}\Big(\tilde{J}_n(\Phi,\Gamma)\Big) = \Omega(n)$, as required. $\qquad\square$

3.5. *Phase transitions in the sparse and thermodynamic regimes.* So far, we have concentrated on the asymptotic behaviour of the expectations of the numbers of different types of subgraphs that appear in the random graph associated with a point process. In this section we shall combine expectations on first and second moment to obtain results about these numbers themselves, looking at probabilities that they are non-zero, as well as L_2 and almost sure results about growth and decay rates. The main theorem of this section is the following:

THEOREM 3.6.   *Let Φ be a stationary point process with almost every-where continuous joint densities and Γ a feasible connected graph for Φ on k vertices.*

1: Let Φ satisfy the assumptions of Theorem 3.1 with $\mu_0(\Phi, \Gamma) > 0$. Let $r_n \to 0$.

   (a) If $n r_n^{d(k-1)} f^k(r_n) \to 0$[4], then $\mathbb{P}\{G_n(\Phi, \Gamma) \geq 1\} \to 0$.
   (b) If Φ is α-negatively associated and $n r_n^{d(k-1)} f^k(r_n) \to \beta$ for some $0 < \beta < \infty$, there there exists a finite C (dependent on the process but not on Γ) for which

$$\lim_{n\to\infty} \mathbb{P}\{J_n(\Phi, \Gamma) \geq 1\} \geq \left[1 + \frac{C}{\beta \mu_0(\Phi, \Gamma)}\right]^{-1}.$$

   (c) If Φ is α-negatively associated and $n r_n^{d(k-1)} f^k(r_n) \to \infty$, then

$$\frac{J_n(\Phi, \Gamma)}{n r_n^{d(k-1)} f^k(r_n)} \overset{L_2}{\to} \mu_0(\Phi, \Gamma).$$

2: Let Φ be a negatively associated point process satisfying the assumptions of Theorem 3.3 with $\gamma_\beta(\Phi, \Gamma) > 0$. Let $r_n^d \to \beta$. Then,

$$\frac{J_n(\Phi, \Gamma)}{n} \overset{L_2}{\to} \gamma_\beta(\Phi, \Gamma).$$

PROOF. The proof for Part 1(a) follows from Markov's inequality and Theorem 3.1. The proof of 1(b) is based on the following second moment bound:

$$
\begin{aligned}
\mathbb{P}\{J_n(\Phi, \Gamma) \geq 1\} &\geq \frac{(\mathbb{E}\{J_n(\Phi, \Gamma)\})^2}{\mathbb{E}\{J_n(\Phi, \Gamma)^2\}} \\
&\geq \frac{(\mathbb{E}\{J_n(\Phi, \Gamma)\})^2}{\mathbb{E}\{G_n(\Phi, \Gamma)^2\}} \\
&\geq \left[\frac{(\mathbb{E}\{G_n(\Phi, \Gamma)\})^2}{(\mathbb{E}\{J_n(\Phi, \Gamma)\})^2} + \frac{\mathsf{Var}(G_n(\Phi, \Gamma))}{(\mathbb{E}\{J_n(\Phi, \Gamma)\})^2}\right]^{-1}.
\end{aligned}
$$

Now, by applying Theorem 3.4, we obtain that there exists a $C > 0$ for which

$$\mathbb{P}\{J_n(\Phi, \Gamma) \geq 1\} \geq \left[\frac{(\mathbb{E}\{G_n(\Phi, \Gamma)\})^2}{(\mathbb{E}\{J_n(\Phi, \Gamma)\})^2} + \frac{C}{\mathbb{E}\{G_n(\Phi, \Gamma)\}}\right]^{-1}.$$

----

[4]Note that neither this assumption nor the one in 1(b) can hold for $k = 1$, as $f^1(r) \equiv 1$. Hence the statements do not say anything in these two cases.

Under the assumptions of 1(b), $\mathbb{E}\{G_n(\Phi,\Gamma)\}$ converges to $\beta\mu_0(\Phi,\Gamma)$, while the first term in the square brackets converges to 1 by Theorem 3.1.

For 1(c), observe that

$$\mathsf{Var}(J_n(\Phi,\Gamma)) \leq \mathsf{Var}(G_n(\Phi,\Gamma)) + 2\mathbb{E}\{G_n(\Phi,\Gamma)\}\,\mathbb{E}\{E_n(\Phi,\Gamma)\} - (\mathbb{E}\{E_n(\Phi,\Gamma)\})^2,$$

where $E_n(\Phi,\Gamma)$ is as defined in the proof of Theorem 3.1. From the proofs of Theorem 3.1 and Theorem 3.4, it follows that

$$\mathsf{Var}(J_n(\Phi,\Gamma)) = O(\mathsf{Var}(G_n(\Phi,\Gamma))) = O(nr_n^{d(k-1)}f^k(r_n)),$$

which completes the proof for this case.

We now prove Part 2 in a similar fashion. In fact, it follows easily from Theorem 3.5 and the relation between J_n and $\tilde{J}_n$ noted in the proof of Theorem 3.3. More specifically, as $n \to \infty$,

$$\mathsf{Var}\left(\frac{\tilde{J}_n(\Phi,\Gamma)}{n}\right) \quad \to \quad 0$$

and

$$\mathbb{E}\left\{\frac{\|J_n(\Phi,\Gamma) - \tilde{J}_n(\Phi,\Gamma)\|}{n}\right\} \quad = \quad \frac{\mathbb{E}\left\{G_n(\Phi_n/\Phi_{(n^{\frac{1}{d}}-(k+1)r_n)^d}, \Gamma)\right\}}{n} \to 0.$$

Thus, we have that $\frac{\tilde{J}_n(\Phi,\Gamma)}{n} \overset{P}{\to} \gamma_\beta(\Phi,\Gamma)$ and $\frac{\|J_n(\Phi,\Gamma)-\tilde{J}_n(\Phi,\Gamma)\|}{n} \overset{P}{\to} 0$ as $n \to \infty$. $\square$

Since J_n is a K_d-Lipschitz functional of counting measures for a constant K_d depending only on the dimension d (see [30, Proof of Theorem 3.15]), the result in (2) above can be strengthened to a.s. convergence for stationary determinantal point processes by using the concentration inequality in [29, Theorem 3.6]. We leave the proof, which is not hard, to the reader.

THEOREM 3.7. *Let Φ be a stationary determinantal point process of unit intensity and Γ a feasible connected graph for Φ on k vertices. Let $r_n^d \to \beta > 0$. Then, for $\gamma_\beta(\Phi,\Gamma)$ as defined in (3.18),*

$$\frac{J_n(\Phi,\Gamma)}{n} \overset{a.s.}{\to} \gamma_\beta(\Phi,\Gamma), \quad as\ n \to \infty.$$

4. Betti numbers of random geometric complexes.   This is really the main section of the paper, giving, as it does, results about the homology of random geometric complexes through their Betti numbers. Despite this, it will turn out that, as mentioned earlier, the hard work for the proofs has already been done in the previous section.

We shall start with a review of the basic topological notions needed to formulate our results, along with an explanation of the connections between Betti numbers of random complexes and component numbers of random geometric graphs. This connection was established and exploited in [18, 19] to extract theorems for Betti numbers from those for the component counts of random geometric graphs.

4.1. *Topological preliminaries.*   Recall that Čech and Vietoris-Rips complexes and their faces were already defined at Definitions 1.1 and 1.2 in the Introduction, and that the dimension of a face σ is $|\sigma| - 1$. Recall also that the edges of the random geometric graph $G(\Phi, r)$ are the 1-dimensional faces of $C(\Phi, r)$ or $R(\Phi, r)$.

Now, however, we require some additional terminology.

The Vietoris-Rips complex $R(\Phi, r)$ is also called the *clique complex* (or *flag complex*) of $G(\Phi, r)$, as the faces are cliques (complete subgraphs) of the $1-$ dimensional faces. Let $H_k(C(\Phi, r))$ and $H_k(R(\Phi, r))$, respectively, denote the k-th simplicial homology groups of the random Čech and Vietoris-Rips complexes. (We shall take our homologies over the field $\mathbb{Z}_2$, but this will not be important.) In this section we shall be concerned with asymptotics for the Betti numbers $\beta_k(C(\Phi_n, r))$ and $\beta_k(R(\Phi_n, r))$, (i.e. the ranks of the homologies) and through them the appearance and disappearance of homology groups.

Next, let P_k be the $(k+1)$-dimensional cross-polytope in $\mathbb{R}^{k+1}$, containing the origin, and defined to be the convex hull of the $2k+2$ points $\{\pm e_i\}$, where $e_1, \ldots, e_{k+1}$ are the standard basis vectors of $\mathbb{R}^{k+1}$. The boundary of P_k, which we denote by O_k, is a k-dimensional simplicial complex, homeotopic to a k-dimensional sphere. In terms of simplicial homology, the existence of subsimplices isomorphic to O_k is the key to understanding k-cycles and so the k-th homology. In fact, from [17, Lemma 5.3] we know that, because of the distance relationships between the vertices of a Vietoris-Rips complex, any non-trivial element of the k-dimensional homology $H_k(R(\Phi, r))$ arises from a subcomplex on at least $2k + 2$ vertices. If it has only $2k + 2$ vertices, then it will be isomorphic to O_k.

Now let $\Gamma_k^j, j = 1, \ldots, n_k$ $(n_k < \infty)$ be an ordering of the different graphs that arise when extending a $(k + 1)$-clique (i.e. a k-dimensional face) to a

minimal connected subgraph on $2k + 3$ vertices. Thus, the Γ_k^j are all graphs on $2k + 3$ vertices, having $\binom{k+1}{2} + k + 2$ edges.

Finally, for a given finite graph Γ, let $\tilde{G}(\Phi, \Gamma)$ denote the number of subgraphs of $G(\Phi, r)$ that are isomorphic to Γ. However, as explained in [3], $\tilde{G}(\Phi_n, \Gamma)$ is a finite linear combination of $G_n(\Phi, \Gamma')$'s with Γ''s being of the same order as Γ.

Then [17, Lemma 5.3] and a dimension bound in [18, (3.1)]) imply the following crucial inequality linking Betti numbers to component and subgraph counts in Vietoris-Rips complexes for $k \geq 1$;

$$(4.20) \qquad J(\Phi, O_k) \leq \beta_k(R(\Phi, r)) \leq J(\Phi, O_k) + \sum_{j=1}^{n_k} \tilde{G}(\Phi, \Gamma_{k,2k+3}^j).$$

A related inequality holds for Čech complexes. Let Γ_k be the graph on k vertices such that any $k - 1$ vertices form a $(k-1)$-clique but Γ_k is not a k-clique. Any collection of vertices X for which $G(X, r) \simeq \Gamma_k$ is said to form an *empty* $(k-1)$-*simplex*. Let Γ_k' be the graph of a $(k-1)$-clique with an extra edge attached to two vertices and Γ_k'' be the graph of a $(k-1)$-clique with a path of length 2 attached to one of the vertices. Both Γ_k' and Γ_k'' are graphs of order $k + 1$. The we have the following combinatorial inequality from [19, (5)] for $k \geq 1$:

$$(4.21) \ \ J(\Phi, \Gamma_{k+2}) \leq \beta_k(C(\Phi, r)) \leq J(\Phi, \Gamma_{k+2}) + G(\Phi, \Gamma_{k+2}') + G(\Phi, \Gamma_{k+2}'').$$

With these combinatorial inequalities in hand, we are now ready to develop limit theorems for the Betti numbers of the random Čech and Vietoris-Rips complexes (Section 4.2) as well as find thresholds for vanishing and non-vanishing of homology groups (Section 4.3).

4.2. *Expectations of Betti numbers.* We return now to the setting of a stationary point process Φ in $\mathbb{R}^d$ and the sequence of finite point processes Φ_n. Our results all follow quite easily from the corresponding limit theorems in Section 3, and we continue to use the notation of that section without further comment.

The underlying heuristic is that in the sparse regime the order is determined by the order of the minimal structure involved in forming homology groups, which is O_k for the random Vietoris-Rips complex and Γ_k for the random Čech complex. Using Theorem 3.1, it is easy to see that these are the leading order terms and that the G and $\tilde{G}$ terms in both (4.20) and (4.21) are, asymptotically, irrelevant. Hence, we have the following result.

THEOREM 4.1 (Sparse regime: $r_n \to 0$). *Let Φ be a stationary point process in $\mathbb{R}^d$ satisfying the assumptions in Theorem 3.1 for all $k \geq 1$. Let $r_n \to 0$. Further, assume that $\mu_0(\Phi, \Gamma_k) > 0$ and $\mu_0(\Phi, O_k) > 0$ for all $k \geq 1$. Then, for all $k \geq 1$,*

$$\lim_{n \to \infty} \frac{\mathbb{E}\{\beta_k(C(\Phi_n, r_n))\}}{nr_n^{d(k+1)} f^{k+2}(r_n)} = \mu_0(\Phi, \Gamma_{k+2}),$$

$$\lim_{n \to \infty} \frac{\mathbb{E}\{\beta_k(R(\Phi_n, r_n))\}}{nr_n^{d(2k+1)} f^{2k+2}(r_n)} = \mu_0(\Phi, O_k).$$

For $k = 0$, we have that

$$\lim_{n \to \infty} \frac{\mathbb{E}\{\beta_0(C(\Phi_n, r_n))\}}{n} = \lim_{n \to \infty} \frac{\mathbb{E}\{\beta_0(R(\Phi_n, r_n))\}}{n} = 1.$$

PROOF. We start with the case $k \geq 1$. From Theorem 3.1, the orders of magnitude of the terms in (4.20) and (4.21) are as follows :

$$\mathbb{E}\{J_n(\Phi, \Gamma_{k+2})\} = \Theta(nr_n^{d(k+1)} f^{k+2}(r_n)),$$

$$\mathbb{E}\left\{G_n(\Phi, \Gamma'_{k+2})\right\} = \Theta(nr_n^{d(k+2)} f^{k+3}(r_n)),$$

$$\mathbb{E}\left\{G_n(\Phi, \Gamma''_{k+2})\right\} = \Theta(nr_n^{d(k+2)} f^{k+3}(r_n)),$$

$$\mathbb{E}\{J_n(\Phi, O_k)\} = \Theta(nr_n^{d(2k+1)} f^{2k+2}(r_n)),$$

$$\mathbb{E}\left\{\tilde{G}(\Phi, \Gamma^j_{k,2k+3})\right\} = \Theta(nr_n^{d(2k+2)} f^{2k+3}(r_n)) \qquad 1 \leq j \leq n_k.$$

Substituting these into (4.20) and (4.21), and using the fact that the limits of $\mathbb{E}\{J_n(\Phi, \Gamma_{k+2})\}$ and $\mathbb{E}\{J_n(\Phi, O_k)\}$ are explicitly known from Theorem 3.1, completes the proof of the theorem.

For the case $k = 0$, the bounds similar to (4.20) on β_0 and a similar argument will give the right asymptotics.   □

Turning now to the thermodynamic regime, and applying the same arguments as in the previous proof, but using Theorem 3.3 in place of Theorem 3.1, we find that all the terms in (4.20) and (4.21) are of order $\Theta(n)$. This leads to the following result.

THEOREM 4.2 (Thermodynamic regime: $r_n^d \to \beta$). *Let Φ be a stationary point process in $\mathbb{R}^d$ satisfying the assumptions in Theorem 3.3 for all $k \geq 1$. Let $r_n^d \to \beta \in (0, \infty)$. Further, assume that $\gamma_\beta(\Phi, \Gamma_k) > 0$ and $\gamma_\beta(\Phi, O_k) > 0$ for all $k \geq 1$. Then, for all $k \geq 0$,*

$$\mathbb{E}\{\beta_k(C(\Phi_n, r_n))\} = \Theta(n), \quad \mathbb{E}\{\beta_k(R(\Phi_n, r_n))\} = \Theta(n).$$

We note without proof that one can obtain ordering results for Betti numbers of $\alpha - w$ ordered point processes in the sparse regime analogous to Corollary 3.2 but not in the thermodynamic regime.

4.3. *Thresholds for homology groups.* Our aim in this subsection is to establish results about the conditions under which different homology groups appear and disappear in the homology of random complexes. We shall need to treat Čech and Vietoris-Rips complexes separately, and start with results on the contractibility of these. We follow these with the key results of the section, on thresholds for the appearance and disappearance of homology groups. These results also show that γ-weakly sub-Poisson point processes have lower vanishing thresholds for given Γ-components. As a corollary to the results on Čech complexes, we also obtain an asymptotic result on the behaviour of the Euler characteristic $\chi(C(\Phi, r)$.

Recall that there are a number of equivalent definitions for Euler characteristic. However, the most natural for us at this point is

$$(4.22) \qquad \chi(C(\Phi, r)) \; := \; \sum_{k \geq 0} (-1)^k \beta_k(C(\Phi, r)).$$

THEOREM 4.3 (Contractibility of Čech complexes).   *Let Φ be a stationary γ-weakly sub-Poisson point process. Then there exists a $C_d > 0$ such that for $r_n \geq C_d (\log n)^{\frac{1}{d}}$, whp $C(\Phi_n, r_n)$ is contractible and $\chi(C(\Phi_n, r_n)) = 1$.*

PROOF. We start with a proof of contractibility, and then show that $\chi(C(\Phi_n, r_n)) = 1$, whp.

As in the proof of contractibility for Poisson Čech complexes in [18, Theorem 6.1], we will show that, for our choice of r_n, the set $\bigcup_{X \in \Phi_n} B_X(r_n/2)$ covers W_n whp. Then, the nerve theorem of [3, Theorem 10.7] implies that the Čech complex is contractible whp.

Let $\mathbb{Z}^d$ be the d-dimensional lattice and let $Q_{z_i}, 1 \leq i \leq N_n$ be an enumeration of the cubes of the scaled lattice $\frac{r_n}{4\sqrt{d}} \mathbb{Z}^d$ that are fully contained within W_n. If every cube contains a point of Φ, then $\bigcup_{X \in \Phi_n} B_X(r_n/2)$ covers W_n. By the union bound,

$$
\begin{aligned}
\mathbb{P}\left\{ W_n \not\subseteq \cup_{X \in \Phi_n} B_X(r_n/2) \right\} \;\; &\leq \;\; \sum_{i=1}^{N_n} \mathbb{P}\left\{ \Phi(Q_{z_i}) = 0 \right\} \\
&\leq \;\; N_n \mathbb{P}\left\{ \Phi_{(1)}(B_O(\frac{r_n}{8\sqrt{d}})) = 0 \right\}
\end{aligned}
$$

$$\leq \quad \frac{(4\sqrt{d})^d n}{r_n^d} e^{-(\frac{r_n}{8\sqrt{d}})^d},$$

where $\Phi_{(1)}$ is the Poisson point process of unit intensity. All that remains is to choose an appropriate $C_d > 0$ to complete the proof of contractibility for general stationary γ-weakly sub-Poisson point processes.

As for the proof of the statement about the Euler characteristic, the following obvious bound suffices.

$$\mathbb{P}\left\{W_n \subset \cup_{X \in \Phi_n} B_X(r_n/2)\right\}$$
$$\leq \mathbb{P}\left\{\beta_0(C(\Phi_n, r_n)) = 1, \beta_k(C(\Phi_n, r_n)) = 0, k \geq 1\right\}$$
$$\leq \mathbb{P}\left\{\chi(C(\Phi_n, r_n)) = 1\right\}.$$

$\square$

With these results in hand, we can now use the bounds (4.20) and (4.21) along with a.s. existence results of Section 3.4 to complete the picture about vanishing and non-vanishing of homology groups of Čech complexes and Vietoris-Rips complexes.

THEOREM 4.4 (Thresholds for Čech complexes). *Let Φ be a stationary point process satisfying the assumptions on its joint intensities $\rho^{(k)}$ as in Theorems 3.1 and 3.3 for all $k \geq 1$. Then the following statements hold :*

1. *Let Φ be a γ-weakly sub-Poisson point process.*

   (a) If
   $$r_n^{d(k+1)} f^{k+2}(r_n) = o(n^{-1}) \ \ or \ \ r_n^d = \omega(\log_n),$$
   then $H_k(C(\Phi_n, r_n)) = 0, k \geq 1$, whp.

   (b) If $r_n^d = \omega(\log_n)$, then $H_0(C(\Phi_n, r_n)) = 1$, whp.

2. *Let Φ be a negatively associated point process. Further assume that $\mu_0(\Phi, \Gamma_k) > 0$ and $\gamma_\beta(\Phi, \Gamma_k) > 0$, both for all $k \geq 1$ and all $\beta > 0$.*

   (a) If
   $$r_n^{d(k+1)} f^{k+2}(r_n) = \omega(n^{-1}) \ \ and \ \ r_n^d = O(1),$$
   then $H_k(C(\Phi_n, r_n)) \neq 0, k \geq 1$, whp.

   (b) If $r_n^d = O(1)$, then $H_0(C(\Phi_n, r_n)) \neq 0$, whp.

In the absence of a contractibility result for the Vietoris-Rips complex, we are unable to estimate the second thresholds, where the homology groups vanish. Thus, we have the following less complete picture for the Vietoris-Rips complex. Since $H_0(C(\Phi_n, r_n)) = H_0(R(\Phi_n, r_n))$, we shall restrict ourselves to only $H_k(R(\Phi_n, r_n))$, $k \geq 1$, in the following theorem.

THEOREM 4.5 (Thresholds for Vietoris-Rips complexes).   *Let Φ be a stationary point process satisfying the assumptions on its joint intensities $\rho^{(k)}$ as in Theorems 3.1 and 3.3 for all $k \geq 1$. Then the following statements hold for $k \geq 1$:*

1. *If*
$$r_n^{d(2k+1)} f^{2k+2}(r_n) = o(n^{-1}),$$

    then $H_k(R(\Phi_n, r_n)) = 0$, whp.

2. *Let Φ be a negatively associated point process. Further assume that $\mu_0(\Phi, O_k) > 0$ and $\gamma_\beta(\Phi, O_k) > 0$, both for all $k \geq 1$ and all $\beta > 0$. If*

$$r_n^{d(2k+1)} f^{2k+1}(r_n) = \omega(n^{-1}) \ \ and \ \ r_n^d = O(1),$$

    then $H_k(R(\Phi_n, r_n)) \neq 0$, whp.

4.4. *Further results for the Ginibre process:.*   Using the special structure of the Ginibre point process, we can improve on the threshold results of the last section. The radius regime for contractibility of Čech complexes over the Ginibre point process and zeros of GEF can be made more precise, as more is known about void probabilities in these cases. Once we have the contractibility results, the upper bounds for vanishing of Betti numbers in this special case can be improved.

THEOREM 4.6 (Contractibility of Čech Complexes).   *Let Φ be the Ginibre point process or zeros of GEF, then there exists a $C_d > 0$ (depending on the point process) such that for $r_n \geq C_d(\log n)^{\frac{1}{4}}$, whp $C(\Phi_n, r_n)$ is contractible. Hence, $H_0(C(\Phi_n, r_n)) = 1$, $H_k(C(\Phi_n, r_n)) = 0$, $k \geq 1$ and $\chi(C(\Phi_n, r_n)) = 1$ whp for $r_n^2 = \omega(\sqrt{logn})$.*

PROOF. The proof follows along similar lines as the proof of Theorem 4.3 except that in this case, the void probabilities are of strictly lower order and so, the radius for contractibility as well. More precisely, we know from [2, Propositions 7.2.1 and Theorem 7.2.3] that for the Ginibre point process and zeros of GEF, $-\log(\mathbb{P}\{\Phi(B_O(r)) = 0\}) = \Theta(r^4)$ as $r \to \infty$. All that remains is to substitute these bounds into the proof of Theorem 4.3 to derive the corresponding results for the Ginibre point process and zeros of GEF. □

For Vietoris-Rips complexes, the contractibility result for the Ginibre point process is a consequence of the upper bounds for the Palm void probabilities. As in the previous subsection, once we have contractibility results, we can obtain upper bounds for vanishing of the Betti numbers as well.

THEOREM 4.7 (Disappearence of homology groups for Vietoris-Rips complexes). *Let Φ be the Ginibre point process. Then there exists a $C_{d,k} > 0$ such that for $r_n \geq C_{d,k}(\log n)^{\frac{1}{4}}$, we have that whp $H_k(R(\Phi_n, r_n)) = 0$, $k \geq 1$.*

The proof uses the discrete Morse theoretic approach (see [14]) similar to that of [18, Theorem 5.1] and the reader is referred to that proof and the appendix in [18] for missing details. As in [18, Theorem 5.1], our proof actually shows topological k-connectivity though we do not state it explicitly to avoid defining further topological notions.

PROOF. As the point process is simple and stationary, index the points in Φ as $X_1, X_2, ...$ such that $\|X_1\| < \|X_2\| < \|X_3\| < ...$. Define V to be the collection of pairs of simplices (V_1, V_2), $V_1 \subset V_2$ with

$$V_1 = [X_{i_1}, \ldots, X_{i_k}] \text{ and } V_2 = [X_{i_0}, X_{i_1}, \ldots, X_{i_k}],$$

where $i_0 < i_1 < ...i_k$. In words, we pair a simplex with another simplex of codimension 1 in the original simplex only if the additional point is closer to the origin than the rest. A simplex that is not in V is said to be a critical simplex. Let C_k be the number of critical k-simplices of V. From discrete Morse theory, we know that $\beta_k(R(\Phi_n, r_n)) \leq C_k$. Thus, we only need to show that $\mathbb{E}\{C_k\} \to 0$ for all $k \geq 1$, for an appropriate choice of radii.

A k-simplex $\mathbf{X} = [X_{i_0}, \ldots, X_{i_k}]$ where $i_0 < i_1 < \ldots < i_k$ is critical only if

$$\Phi_n(\cap_{j=0}^k B_{X_{i_j}}(r) \cap B_O(\|X_{i_0}\|)) \;=\; \{X_{i_0}\}.$$

Hence, using Campbell-Mecke formula for the first inequality, then using [18, Lemma 5.3] – i.e. for a critical k-simplex as above, there exists an $\epsilon_d > 0$ and $x \in \mathbb{R}^d$ such that

$$B_x(\epsilon_d) \;\subset\; \bigcap_{j=0}^k B_{X_{i_j}}(r) \cap B_O(\|X_{i_0}\|)$$

– and Lemma 6.1 for the second inequality and finally $\rho^{(k)} \leq 1$ for the last inequality, we find that

$$
\begin{aligned}
\mathbb{E}\{C_k\} \;\leq\; & \int_{W_n^{k+1}} 1[\mathbf{x} \text{ is a simplex}]1[\|x_{i0}\| < \|x_{i_1}\| < \ldots < \|x_{i_k}\|] \\
& \times \mathbb{P}^!_{\mathbf{x}}\{\Phi_n(\cap_{j=0}^k B_{x_{i_j}}(r) \cap B_O(\|x_{i_0}\|)) = 0\}\rho^{(k)}(\mathbf{x})\, d\mathbf{x}. \\
\leq\; & \exp\{(k+1)(\epsilon_d r)^2 - (\epsilon_d r)^4(\frac{1}{4} + o(1))\} \int_{W_n \times B_{x_{io}}(r)^k} \rho^{(k)}(\mathbf{x})\, d\mathbf{x}.
\end{aligned}
$$

$$\leq \quad nr^{2k}\exp\{k(\epsilon_d r)^2 - (\epsilon_d r)^4(\frac{1}{4} + o(1))\}.$$

It is easy to see that there exists a constant $C_{d,k} > 0$ such that $\mathbb{E}\{C_k\} \to 0$ for $r_n \geq C_{d,k}(\log n)^{\frac{1}{4}}$, and so we are done.   □

5. Morse theory for random geometric complexes. Our aim in this section is to present a collection of results looking at random geometric complexes from the point of view of Morse theory. This was done in considerable detail for Poisson and iid point processes in [7], where it was shown that a Morse theoretic approach can give an intrinsically richer set of results than that obtained by attacking homology directly, and that the Morse theoretic results have, as usual, implications about Betti numbers.

We do not intend to give full proofs here, but rather to set things up in such a way that parallels between the structures that have appeared in previous sections and those that are natural to the Morse theoretic approach become clear, and it becomes 'obvious' what the Morse theoretic results will be. Full proofs would require considerable more space, but would add little in terms of insight. We note, however, that this does not make the proofs of [7] in any way redundant. On the one hand, the results there go beyond what we have here (albeit only for the Poisson and iid cases) and it is their existence that allows us to be certain that the parallels work properly.

We start with some definitions and a quick description of the Morse theoretic setting.

5.1. *Morse Theory.* Morse theory for geometric complexes is based on the distance function, $d_\Phi : \mathbb{R}^d \to \mathbb{R}_+$, defined by

$$d_\Phi(x) := \min_{X \in \Phi} \|x - X\|, \quad x \in \mathbb{R}^d.$$

Note that while classical Morse theory deals with smooth functions, the distance function is piecewise linear, but non-differentiable along subspaces. The extension to the distance function of classical Morse theory is discussed in detail in [7], based on the definitions and results in [15], and we shall adopt the same approach. The main difference between smooth Morse theory and that based on the distance function lies in the definition of the indices of critical points.

Critical points of index 0 of the distance function are the points where $d_\Phi = 0$, which are local and global minima, and are the points of Φ. For higher indices, define the critical points as follows : A point $c \in \mathbb{R}^d$ is said to be a *critical point* with index $1 \leq k \leq d$ if there exists a collection of points $\mathbf{X} = \{X_1, \ldots, X_{k+1}\} \subset \Phi^{(k+1)}$ such that the following conditions hold :

1. $d_\Phi(c) = \|c - X_i\|$ for all $1 \le i \le k+1$ and $d_\Phi(c) < \|c - Y\|$ for all $Y \in \Phi \setminus \mathbf{X}$.
2. The points $X_i, 1 \le i \le k+1$ lie in general position, viz. they do not lie in a $(k-1)$-dimensional affine space.
3. $c \in conv^o(\mathbf{X})$, where $conv^o(\mathbf{X})$ denotes the interior of the convex hull formed by the points of $\mathbf{X}$.

Let $C(\mathbf{X})$ denote the center of the unique $(k-1)$-dimensional sphere (if it exists) containing the points of $\mathbf{X} \in \Phi^{(k+1)}$ and $R(\mathbf{X})$ be the radius of the ball. The conditions in the definition of critical points can be reduced to the following more workable conditions (see [7, Lemma 2.2]). A set of points $\mathbf{X} \in \Phi^{(k+1)}$ in general position generates an index k critical point if, and only if,

$$C(\mathbf{X}) \in conv^o(\mathbf{X}) \quad \text{and} \quad \Phi(B_{C(\mathbf{X})}(R(\mathbf{X}))) = 0.$$

Our interest lies in critical points which are at most at a distance r from Φ, viz. those for which $d_\Phi(c) \le r$, or, equivalently $R(\mathbf{X}) \le r$. The reason for this lies in the simple fact that

$$d_\Phi^{-1}([0, r]) \;=\; \bigcup_{x \in \Phi} B_x(r),$$

and, as we already noted earlier, by the nerve theorem this is homotopy equivalent to the Čech compilex $C(\Phi, r)$.

The following indicator functions will be required to draw the analogy between counting critical points and counting components of random geometric graphs. For $\mathbf{X} \in \Phi^{(k+1)}$, define

$$h(\mathbf{X}) := \mathbb{1}[C(\mathbf{X}) \in conv^o(\mathbf{X})], \quad h_r(\mathbf{X}) := \mathbb{1}[C(\mathbf{X}) \in conv^o(\mathbf{X})]\mathbb{1}[R(\mathbf{X}) \le r].$$

Note that these functions are translation and scale invariant, as were the h_Γ functions defined for the subgraph and component counts in Section 3, viz. for all $x \in \mathbb{R}^d$ and $\mathbf{y} = (0, y_1, \ldots, y_k) \in \mathbb{R}^{d(k+1)}$,

$$h_r(x, x + ry_1, \ldots, x + ry_k) = h_1(\mathbf{y}).$$

This was the key property of h_Γ used to derive asymptotics for component counts. Thus, once we manage to represent the numbers of critical points as counting statistics of h_r, the analogy with component counts is made. To this end, let $N_k(\Phi, r)$ be the number of critical points of index k for the distance function d_Φ that are at most at a distance r from Φ. Then,

$$(5.23) \qquad N_k(\Phi, r) \;=\; \sum_{\mathbf{X} \in \Phi^{(k+1)}} h_r(\mathbf{X})\mathbb{1}[\Phi(B_{C(\mathbf{X})}(R(\mathbf{X}))) = 0]$$

The similarity between the expression for $N_k(\Phi_n, r_n)$ and J_n (cf. (3.11)) should convince the reader that the method of proof used for component counts will also suffice for a derivation of the asymptotics of Morse critical points. Although the void indicator term is slightly different, we can use the fact that $R(\mathbf{X}) \leq r$ for $h_r(\mathbf{X}) = 1$ to apply the techniques of Section 3 with only minor changes.

5.2. *Limit theorems for expected numbers of critical points.* As in previous sections, we shall give results for the sparse and thermodynamic regimes separately. In the Betti number results, in the sparse regime $(r_n \to 0)$ the scaling factor of n for J_n (see Theorem 3.1) arose from the translation invariance of h_Γ and Φ. The factor of $r_n^{d(k-1)}$ was due to the scale invariance of h_Γ, and the factor of $f^k(r_n)$ came from the scaling of the joint intensities $\rho^{(k)}$. Since h_r is also translation and scale invariant, we work under the same assumptions on Φ as in Theorems 3.1 and 3.4 with corresponding conditions h_r in order to obtain asymptotics for expected number of critical points of the distance function. The corresponding result is as follows:

THEOREM 5.1 (Sparse regime).   *Let Φ be a stationary point process in $\mathbb{R}^d$ satisfying the assumptions of Theorem 3.1 for all $1 \leq k \leq (d+1)$. Let $r_n \to 0$ and $\mathbf{y} = (0, y_1, y_2, \ldots, y_k)$. Then, for $0 \leq k \leq d$,*

$$\lim_{n \to \infty} \frac{\mathbb{E}\{N_k(\Phi_n, r_n\}}{n r_n^{dk} f^{k+1}(r_n)} \quad = \quad \nu_k(\Phi, 0)$$

$$:= \quad \begin{cases} 1 & k = 0, \\ \frac{1}{(k+1)!} \int_{\mathbb{R}^{dk}} h_1(\mathbf{y}) g_\rho^{k+1}(\mathbf{y}) \, d\mathbf{y} & 1 \leq k \leq d. \end{cases}$$

Further, $\mathsf{Var}(N_k(\Phi_n, r_n)) = O(\mathbb{E}\{N_k(\Phi_n, r_n\})$ for negatively associated point processes and $\mathsf{Var}(N_k(\Phi_n, r_n)) = \Omega(\mathbb{E}\{N_k(\Phi_n, r_n\})$ for associated point processes.

One point that is deserving of additional comment for the proof is that, as in Theorem 3.1, we can omit the void probability term in the limit by the following reasoning: Since $R(\mathbf{y}) \leq r$ if $h_r(\mathbf{y}) = 1$, $\mathbf{y} = (0, y_1, \ldots, y_k)$, we have that whenever $h_{r_n}(\mathbf{y}) = 1$,

$$\left\{ \Phi(B_{C(r_n^{\frac{1}{d}} \mathbf{y})}(r_n^{\frac{1}{d}})) = 0 \right\} \subset \left\{ \Phi(B_{C(r_n^{\frac{1}{d}} \mathbf{y})}(r_n^{\frac{1}{d}} R(\mathbf{y})) = 0 \right\},$$

and the probability of the left event here (and hence the right as well) tends to 1. This follows from similar arguments to those in the proof of Theorem 3.1.

Turning now to the thermodynamic regime, we saw in Theorem 3.3 that the sole scaling factor of n for component counts is due to the translation invariance of h_Γ and Φ. The same remains true for mean numbers of critical points.

THEOREM 5.2 (Thermodynamic regime : $r_n^d \to \beta$.). *Let Φ be a stationary point process in $\mathbb{R}^d$ satisfying the assumptions of Theorem 3.3 for all $1 \leq k \leq (d+1)$. Let $r_n^d \to \beta \in (0, \infty)$ and $\mathbf{y} = (0, y_1, y_2, \ldots, y_k)$. Then, for $0 \leq k \leq d$,*

$$\lim_{n \to \infty} \frac{\mathbb{E}\{N_k(\Phi_n, r_n\}}{n}$$
$$= \nu_k(\Phi, \beta)$$
$$:= \begin{cases} \mathbb{P}_O^!(\Phi(B_O(\beta^{\frac{1}{d}} R(\mathbf{y}))) = 0) & \text{if} \quad k = 0, \\ \frac{\beta^k}{(k+1)!} \int_{\mathbb{R}^{dk}} h_1(\mathbf{y}) \mathbb{P}_{\beta^{\frac{1}{d}}\mathbf{y}}^! (\Phi(B_{C(\beta^{\frac{1}{d}}\mathbf{y})}(\beta^{\frac{1}{d}} R(\mathbf{y}))) = 0)\rho^{(k)}(\beta^{\frac{1}{d}}\mathbf{y}) \, d\mathbf{y} \\ & \text{if} \quad 1 \leq k \leq d. \end{cases}$$

Further, assume that Φ is also a negatively associated point process such that

$$\mathbb{P}\left\{\Phi(B_{C(\mathbf{x})}(\beta^{\frac{1}{d}})) = 0\right\} > 0$$

for a.e. $\mathbf{x} = (0, x_1, \ldots, x_k) \in B_0(3\beta^{\frac{1}{d}})^{k+1}$, and for all $1 \leq k \leq d$. Then $\nu_k(\Phi, \beta) > 0$ for all $0 \leq k \leq d$.
Also, $\mathsf{Var}(N_k(\Phi_n, r_n)) = O(\mathbb{E}\{N_k(\Phi_n, r_n\})$ for negatively associated point processes and $\mathsf{Var}(N_k(\Phi_n, r_n)) = \Omega(\mathbb{E}\{N_k(\Phi_n, r_n\})$ for associated point processes.

As previously, the void probability needs some attention. In this case, to show its positivity, we again use the fact that $R(\mathbf{y}) \leq 1$ if $h_1(\mathbf{y}) = 1$ and hence, whenever $h_1(\mathbf{y}) = 1$,

$$\{\Phi(B_{C(\beta^{\frac{1}{d}}\mathbf{y})}(\beta^{\frac{1}{d}})) = 0\} \subset \{\Phi(B_{C(\beta^{\frac{1}{d}}\mathbf{y})}(\beta^{\frac{1}{d}} R(\mathbf{y}))) = 0\}.$$

The positivity of the first event under Palm probability is guaranteed by our assumption via Lemma 2.1.

Finally, we turn to a result about Euler characteristics that is not accessible from the non-Morse theory. We already defined the Euler characteristic in terms of Betti numbers at (4.22), and showed in Theorem 4.3 that, in the connectivity regime, it is 1 with high probability. However, taking an

alternative, but equivalent, definition via numbers of Morse critical points, we can deduce its L_1 asymptotics in the sparse and thermodynamic regimes as a corollary of the previous results in this section. The alternative definition, which is more amenable to computations due to the bounded number of terms in the following sum, is

$$\chi(C(\Phi, r)) \ := \ \sum_{k=0}^{d} (-1)^k N_k(\Phi, r).$$

THEOREM 5.3.   *Let Φ be a stationary point process in $\mathbb{R}^d$ satisfying the assumptions of Propositions 3.1 and 3.3 for all $1 \leq k \leq (d+1)$.*

(i) *If $r_n \to 0$, then*
$$n^{-1}\mathbb{E}\{\chi(C(\Phi_n, r_n))\} \to 1.$$

(ii) *If $r_n^d \to \beta \in (0, \infty)$, then*

$$n^{-1}\mathbb{E}\{\chi(C(\Phi_n, r_n))\} \to 1 + \sum_{k=1}^{d}(-1)^k \nu_k(\Phi, \beta).$$

(iii) *If Φ is also a negatively associated point process, then the above convergences also hold in the L_2-norm.*

To prove the Part (iii) of the theorem, we need variance bounds, which is why we require the additional assumption of negative association. For example, in the sparse regime, we have the following bound via the Cauchy-Schwarz inequality :

$$
\begin{aligned}
\mathbb{E}\left\{ \left\| \frac{\chi(C(\Phi_n, r_n))}{n} - 1 \right\|^2 \right\} \ &= \ \mathbb{E}\left\{ \left\| (\frac{\Phi(W_n)}{n} - 1) + \sum_{k=1}^{d}(-1)^k \frac{N_k(\Phi, r)}{n} \right\|^2 \right\} \\
&\leq \ d\left(\frac{\mathsf{Var}(\Phi(W_n))}{n^2} + \sum_{k=1}^{d} \frac{\mathbb{E}\{N_k(\Phi_n, r_n)^2\}}{n^2} \right) \\
&= \ d\left(\frac{\mathsf{Var}(\Phi(W_n))}{n^2} + \sum_{k=1}^{d} \frac{\mathsf{Var}(N_k(\Phi_n, r_n))}{n^2} \right. \\
&\qquad \left. + \sum_{k=1}^{d} \frac{(\mathbb{E}\{N_k(\Phi_n, r_n)\})^2}{n^2} \right).
\end{aligned}
$$

The L_2 convergence follows once it is noted that all the terms on right hand side converge to 0 due to the variance bounds proven for negatively

associated point processes. A slight modification of this argument handles the thermodynamic regime as well.

6. Appendix. In this section, we prove the result about Palm void probabilities of Ginibre point process that is used in the proof of Theorem 4.7. The proof is due to Manjunath Krishnapur.

LEMMA 6.1. *Let $D = B_0(r) \subset \mathbb{R}^2$ for some $r > 0$ and Φ be the Ginibre point process. Then for $k \geq 1$ and $\mathbf{x} \in \mathbb{R}^{2k}$,*

$$\mathbb{P}^!_{\mathbf{x}}\{\Phi(D) = 0\} \leq \exp\{kr^2\}\mathbb{P}\{\Phi(D) = 0\} = \exp\{kr^2 - r^4(\frac{1}{4} + o(1))\}.$$

PROOF. We shall prove the result for $k = 1$. The proof for the general case then follows by a recursive application of the same argument.

Let $\mathcal{K}_D$ be the restriction to D of the integral operator $\mathcal{K}$ corresponding to Ginibre point process. Since the Palm process of the Ginibre point process is also a determinantal point process, let $\mathcal{L}_D$ be the integral operator corresponding to the Palm point process restricted to D. Let $\lambda_i, i = 1, 2, \ldots$ and $\mu_i, i = 1, 2, \ldots$ be the eigenvalues of $\mathcal{K}_D$ and $\mathcal{L}_D$ respectively. From [32, Theorem 6.5], we know that $\mathcal{K}_D - \mathcal{L}_D$ has rank 1, and hence, by a generalisation of Cauchy's interlacement theorem, the respective eigenvalues are interlaced with $\lambda_i \geq \mu_i \geq \lambda_{i+1}$ for $i = 1, 2, \ldots$.

Now, consider the case $k = 1$ and we have the following inequality :

$$\mathbb{P}^!_{\mathbf{x}}\{\Phi(D) = 0\} = \prod_{i \geq 1}(1 - \mu_i) \leq \prod_{i \geq 2}(1 - \lambda_i) = (1 - \lambda_1)^{-1}\mathbb{P}\{\Phi(D) = 0\},$$

where the two equalities are due to Theorem [2, 4.5.3] and the inequality is due to the generalisation of Cauchy's interlacement theorem described above. Now using Proposition [2, Proposition 7.2.1] to bound $\mathbb{P}\{\Phi(D) = 0\}$ and from the fact that $1 - \lambda_1 = \mathbb{P}\{EXP(1) > r^2\} = \exp\{-r^2\}$ (see proof of [2, Theorem 4.7.1]), where $EXP(1)$ is the exponential random variable with mean 1, we have the desired inequality for the case $k = 1$. □

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