Random Structures and Algorithms

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Contents of talk

(a) Random Discrete Structures

(b) Random Instances of the TSP in the unit square $[0,1]^2$

(c) The Random Graphs $G_{n,m}$ and $G_{n,p}$.

  (1) Evolution
  (2) Chromatic number
  (3) Matchings
  (4) Hamilton cycles

(d) Randomly edge weighted graphs

  1. Minimum Spanning Tree
  2. Shortest Paths
  3. 3-Dimensional Assignment Problem
  4. Random Instances of the TSP with independent costs

(e) Random k-SAT

(f) Open Problems

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This talk will be about Probabilistic Combinatorics/Analysis of Algorithms (average case).

Random Discrete Structures

There is unfortunately no time to discuss the Probabilistic Method where one uses probabilistic arguments to prove the existence of certain mathematical entities.

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TSP in the Unit Square

It is a feature of modern computation that many of the problems we would like to solve seem hard, in some well-defined sense e.g. NP-hard. As such any algorithm is likely to take a large amount of time on some problems.

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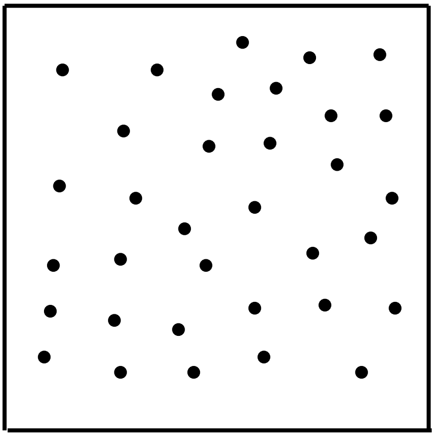
In practise, typical problems are often easy to satisfactorily solve.

Karp (1977) pioneered the idea of finding algorithms that work well on instances drawn from some natural probability distribution. He focussed first on the Travelling Salesperson Problem (TSP).

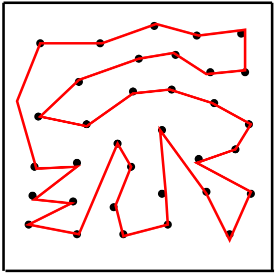
TSP in the Unit Square

Let $\mathcal{X} = X_1, X_2, \ldots, X_n$ be n points chosen independently and uniformly from $[0, 1]^2$.



TSP in the Unit Square

$X_1, X_2, \ldots, X_n$ are independently chosen, uniformly from $[0,1]^2$.



Let Z be the minimum total length of a closed path (tour) through $X_1, X_2, \ldots, X_n$.

We consider the likely value of Z as $n \to \infty$.

Theorem (Beardwood, Halton and Hammersley (1959))

There exists an absolute constant $\beta > 0$ such

$$\frac{Z}{n^{1/2}} \to \beta \text{ with probability 1}$$

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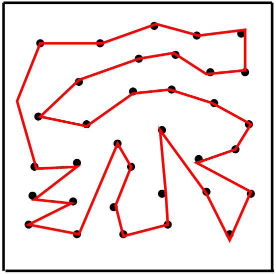

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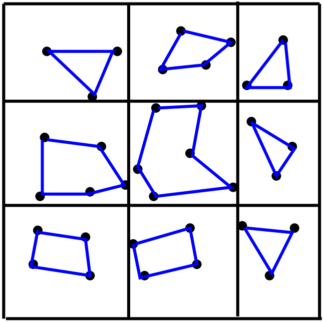
We consider the likely value of Z as $n \to \infty$.

The precise value of β is unkown to this day.

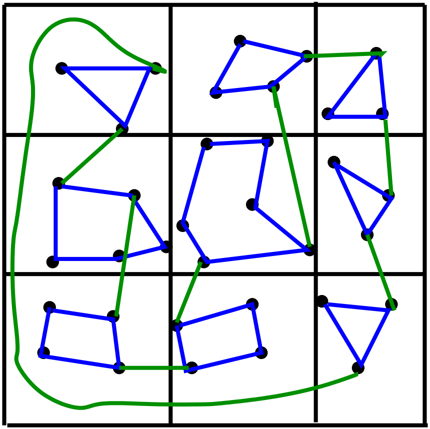
Karp (1977) described a heuristic that runs in polynomial time and w.h.p. produces a tour of length $Z + o(n^{1/2})$.



Sub-square size $(\log n/n)^{1/2}$

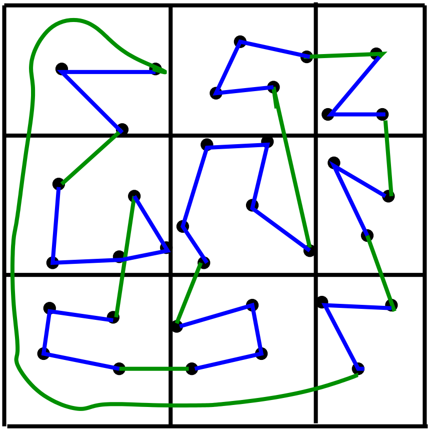
Solve the individual problems in each sub-square.

TSP in the Unit Square



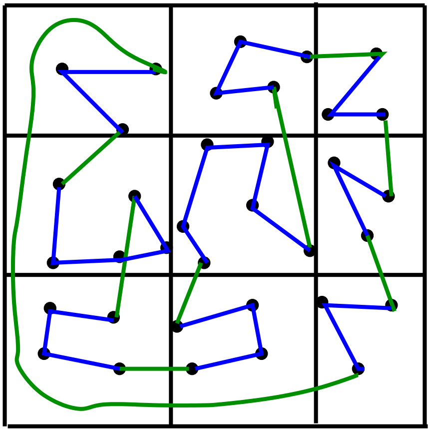
Connect up the smaller tours as shown, by green edges.

TSP in the Unit Square



Connect up the smaller tours as shown, by green edges.
Convert to tour by deleting excess edges.

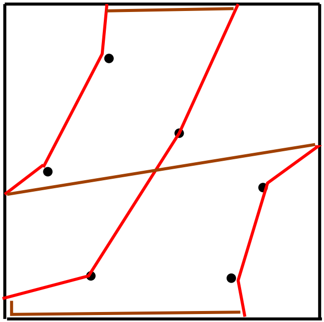
TSP in the Unit Square



The total length of the green edges is $O(n^{1/2}/L) = o(n^{1/2})$.

Travelling Salesman in the Unit Square:

Single Sub-Square



Red edges from optimal tour through all n points.
Red plus Brown edges at least as long as the one found in the sub-square by the algorithm.
Total length of brown edges is $O(n/L^2) \times Ln^{-1/2} = o(n^{1/2})$.

TSP in the Unit Square

Theorem (Karp (1977))

There is a polynomial time algorithm that w.h.p. finds a tour of length $(1+o(1))\beta n^{1/2}$.

Here w.h.p. (with high probability) means with probability $1-o(1)$ as $n \to \infty$.

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Note that Papadimitriou (1977) showed that the TSP restricted to Euclidean instances is still NP-hard.

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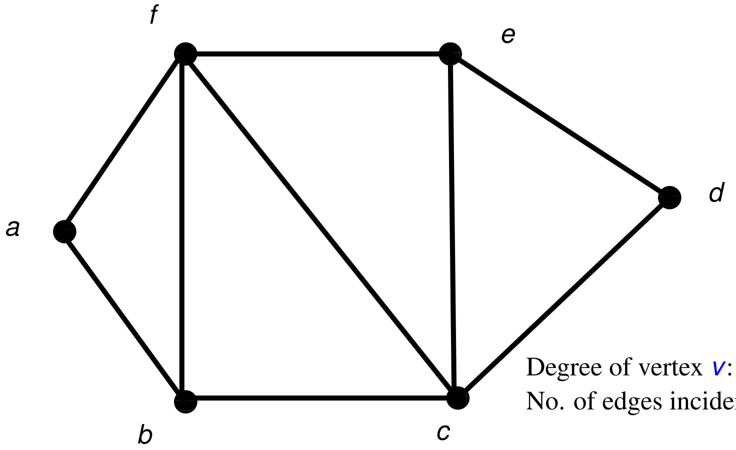
$G_{n,m}$ and $G_{n,p}$

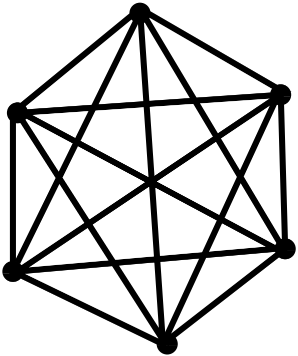


Degree of vertex v:
No. of edges incident with v

Graph $G = (V, E)$
Vertices $V = \{a, b, c, d, e, f\}$
Edges $E = \{\{a, b\}, \{a, f\}, \ldots, \{e, f\}\}$

$G_{n,m}$ and $G_{n,p}$

The complete graph K_n has vertex set $[n] = \{1, 2, \ldots, n\}$ and edge set $\binom{[n]}{2}$.



K_6

Choosing a graph at random

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$G_{n,m}$: Vertex set $[n]$ and m random edges.

$G_{n,p}$: Each edge e of the complete graph K_n is included independently with probability $p = p(n)$.

W.h.p. $G_{n,p}$ has $\sim \binom{n}{2}p$ edges, provided $\binom{n}{2}p \to \infty$

$p = 1/2$, each subgraph of K_n is equally likely.

If $m \sim \binom{n}{2}p$ then $G_{n,p}$ and $G_{n,m}$ have "similar" properties.

W.h.p. means with probability 1-o(1) as $n \to \infty$.

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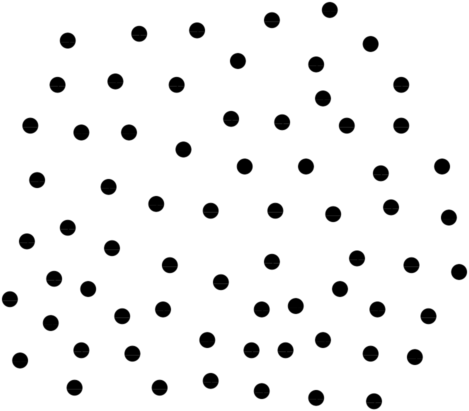
The evolution of a random graph

Graph process $G_0, G_1, \ldots$ where G_{i+1} is G_i plus a random edge.



In the beginning

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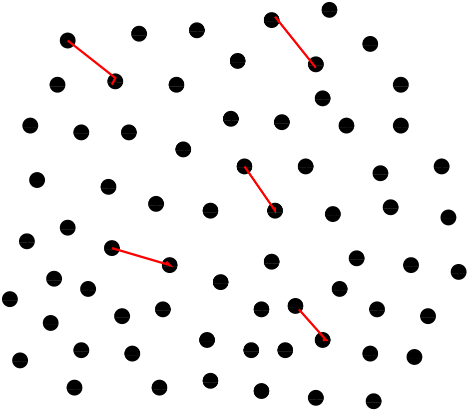

$m = o(n^{1/2})$

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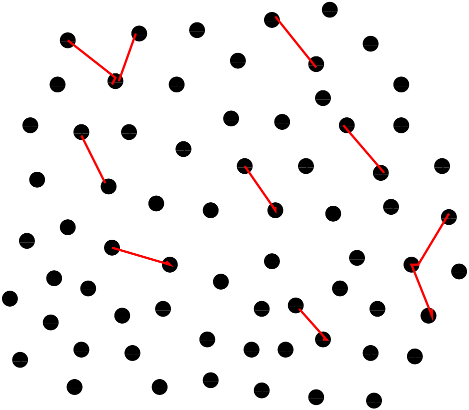

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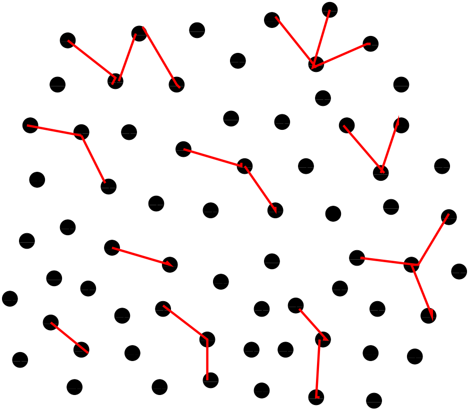

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Erdős and Rényi (1960)

m	Structure of $G_{n,m}$ w.h.p.
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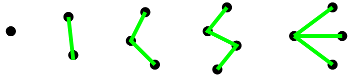

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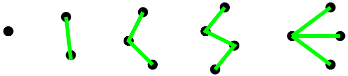

$\omega(n^{\frac{k-1}{k}})$	Components are trees with $1 \leq j \leq k+1$ vertices. Each possible such tree appears.

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$\frac{1}{2}n$	Fascinating. Maximum component size order $n^{2/3}$. Has subsequently been the subject of more intensive study e.g. Janson, Knuth, Łuczak and Pittel (1993).

The evolution of a random graph

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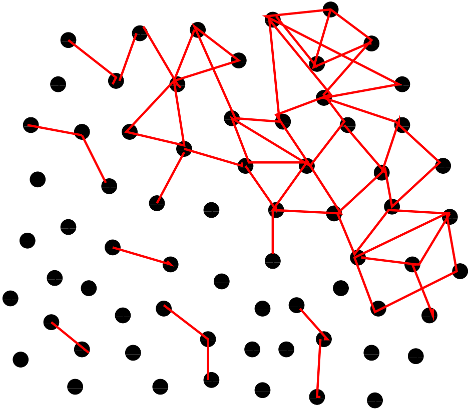
$\frac{1}{2}cn$ $c > 1$	Unique giant component of size $\sim \gamma(c)n$. Remainder almost all trees. Second largest component of size $O(\log n)$

$\gamma(c)$ is the probability that a branching process where each particle has a Poisson, mean c, number of descendants, does not go extinct.

The evolution of a random graph



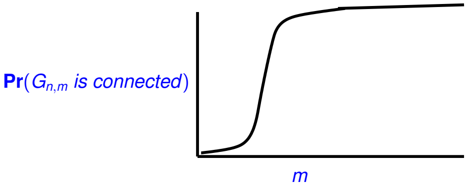
$m = cn/2,\ c > 1$

The evolution of a random graph

Theorem (Erdős and Rényi (1959))

$m = \frac{1}{2}n(\log n + c_n)$

$$\lim_{n\to\infty} \mathbf{Pr}(G_{n,m} \textit{ is connected}) = \begin{cases} 0 & c_n \to -\infty \\ e^{-e^{-c}} & c_n \to c \\ 1 & c_n \to +\infty \end{cases} = \lim_{n\to\infty} \mathbf{Pr}(\delta(G_{n,m}) \geq 1)$$



Notice the sharp transition from disconnected to connected.

Connectivity threshold

$$p = (1+\epsilon)\frac{\log n}{n}, \quad \left(m = \frac{1+\epsilon}{2} n \log n.\right)$$

X_k = number of k-components, $1 \le k \le n/2$.
$X = X_1 + X_2 + \cdots + X_{n/2}$
$G_{n,p}$ is connected iff $X = 0$.

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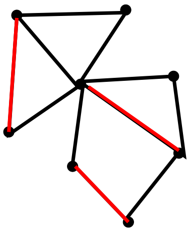
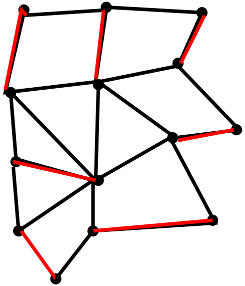
$$\begin{aligned}
\mathbf{Pr}(X \neq 0) &\leq \mathbf{E}(X) \\
&\leq \sum_{k=1}^{n/2} \binom{n}{k} k^{k-2} p^{k-1} (1-p)^{k(n-k)} \\
&\leq \frac{n}{\log n} \sum_{k=1}^{n/2} \left(\frac{e \log n}{n^{(1+\epsilon)(1-k/n)}} \right)^k \\
&\to 0.
\end{aligned}$$

Evolution: Matchings

A matching in a graph G is a set of vertex disjoint edges. The matching is perfect if every vertex is covered by an edge of the matching.



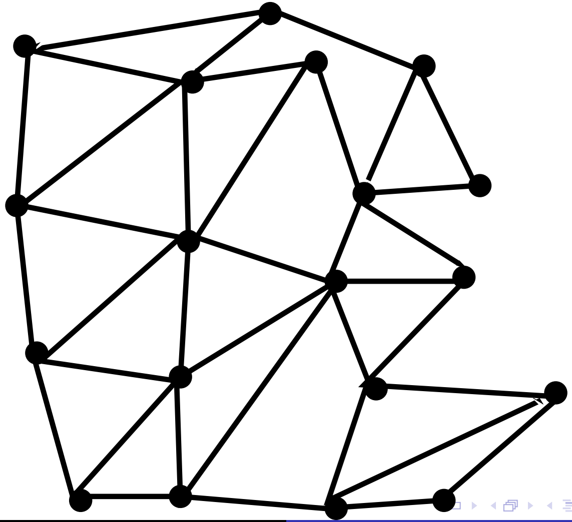
Matching



Perfect Matching

Evolution: Hamilton cycles

A Hamilton cycle in a graph G is a cycle that passes through each vertex exactly once.



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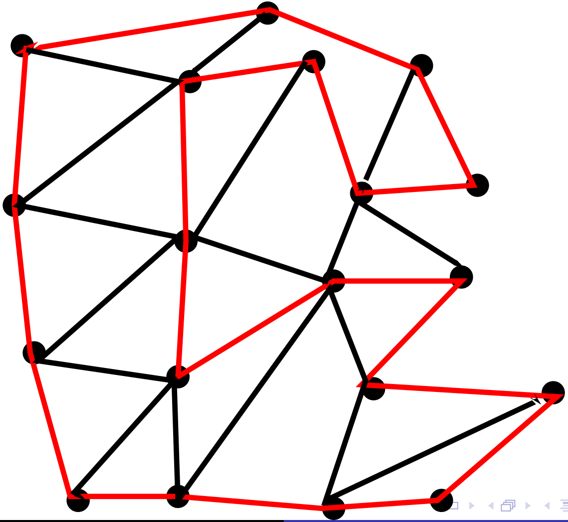

Evolution: Hitting Times

Consider $G_0, G_1, \ldots, G_m, \ldots,$: G_{i+1} is G_i plus a random edge. Let m_k denote the minimum m for which the minimum vertex degree $\delta(G_m) \geq k$.

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W.h.p. m_1 is the "time" when G_m first becomes connected.

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Theorem (Erdős and Rényi (1966))

W.h.p. m_1 is the "time" when G_m first has a perfect matching.

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Theorem (Ajtai, Komlós, Szemerédi (1985), Bollobás (1984))

W.h.p. m_2 *is the "time" when* G_m *first has a Hamilton cycle.*

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Theorem (Cooper and Frieze (1989))

W.h.p. at "time" m_2, G_m has $(\log n)^{n-o(n)}$ Hamilton cycles.

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Theorem (Glebov and Krivelevich (2013))

W.h.p. at "time" m_2, G_m has $n!p^n e^{-o(n)}$ Hamilton cycles.

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Theorem (Bollobás and Frieze (1985))

W.h.p. at "time" $m_k, k = O(1)$, G_m has $\lfloor k/2 \rfloor$ disjoint Hamilton cycles plus a disjoint perfect matching if k is odd– Property $\mathcal{A}_k$.

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Theorem (Knox, Kühn and Osthus (2012))

W.h.p. G_m has property $\mathcal{A}_\delta$ for $n \log^{50} n \leq m \leq \binom{n}{2} - o(n^2)$.

Theorem (Krivelevich and Samotij (2012))

W.h.p. G_m has property $\mathcal{A}_\delta$ for $\frac{1}{2} n \log n \leq m \leq n^{1+\epsilon}$.

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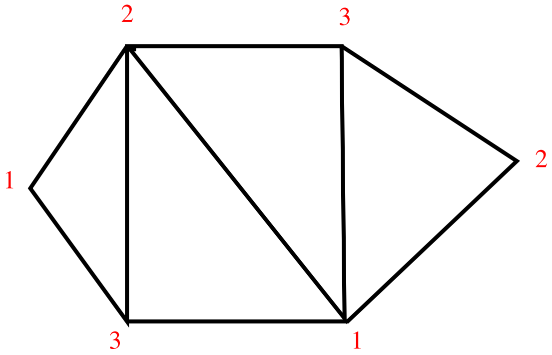
  4. Random Instances of the TSP with independent costs

(e) Random k-SAT

(f) Open Problems

Chromatic Number

A proper k-coloring of a graph G is a map $f : V \rightarrow [k]$ such that if $\{v, w\}$ is an edge of G then $f(v) \neq f(w)$.



The chromatic number $\chi(G)$ is the smallest k for which there is a proper k-coloring.

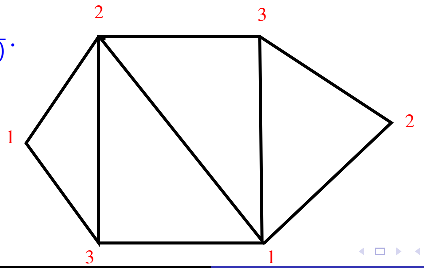
A set of vertices $S \subseteq V$ is independent if $v, w \in S$ implies that $\{v, w\}$ is not an edge.

In a proper k-coloring, each color class is an independent set.

The independence number $\alpha(G)$ is the size of the largest independent set.

Thus

$$\chi(G) \geq \frac{|V|}{\alpha(G)}.$$



Theorem (Matula (1970))

W.h.p. the maximum size $\alpha(G_{n,1/2})$ *of an independent set is*

$$2\log_2 n - 2\log_2\log_2 n + O(1).$$

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Start with $I = \{1\}$.
Repeatedly add $v \notin I$ that is not adjacent to I, until no such v can be found.

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After k successful steps, $\mathbf{E}(\#\text{ choices for } v) \sim n2^{-k}$.

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It may not be possible to find such an independent set in polynomial time w.h.p.

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It may even be NP-hard to find such a coloring in polynomial time w.h.p.

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Let $Z = Z(X_1,\ldots,X_N)$ where $X_1,\ldots,X_N$ are independent. Suppose that changing one X_i only changes Z by ≤ 1. Then

$$\mathbf{Pr}(|Z - \mathbf{E}(Z)| \geq t) \leq e^{-2t^2/N}.$$

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Further inequalities by Talagrand (1995) and Kim and Vu (2000) have been extremely useful.

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$$\mathbf{E}(Z) = n^{2-o(1)} \text{ and changing one edge changes } Z \text{ by } \leq 1$$

So,

$$\mathbf{Pr}(\exists S \subseteq [n]: \ |S| \geq \frac{n}{(\log_2 n)^2} \text{ and } S \text{ doesn't contain a } (2-o(1))\log_2 n \text{ independent set}) \leq 2^n e^{-n^{2-o(1)}} = o(1).$$

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So, we color $G_{n,1/2}$ with color classes of size $\sim 2\log_2 n$ until there are $\leq n/(\log_2 n)^2$ vertices uncolored and then give each remaining vertex a new color.

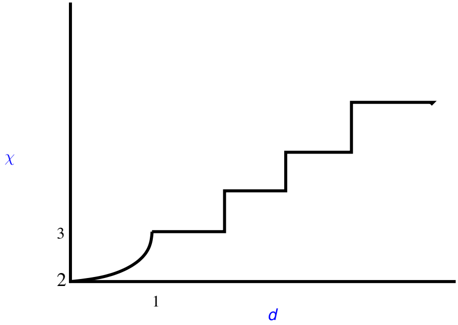
Chromatic Number

There has recently been a lot of research concerning the chromatic number of sparse random graphs viz. $G_{n,p},\ p = d/n$ where $d = O(1)$.

Conjecture: There exists a sequence $d_k : k \geq 2$ such that w.h.p.

$$\chi(G_{n,d/n}) = k \text{ for } d_{k-1} < d < d_k.$$

Friedgut (1999), Achlioptas and Friedgut (1999) came close to proving this.



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Theorem (Łuczak (1991))

W.h.p. $\chi(G_{n,d/n})$ takes one of two values.

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Surprisingly, using a second moment method we get

Theorem (Achlioptas and Naor (2005))

Let k_d be the smallest integer $k \geq 2$ such that $d < 2k \log k$ then w.h.p. $\chi(G_{n,d/n}) \in \{k_d, k_d + 1\}$.

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If X denotes the number of k-colorings of $G_{n,d/n}$ then

$$\mathbf{Pr}(X > 0) \geq \frac{\mathbf{E}(X)^2}{\mathbf{E}(X^2)} = \Omega(1)$$

for $d < 2(k-1)\log(k-1)$.
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The idea is straightforward. The difficulty lies in estimating $\mathbf{E}(X^2)$.

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Now use results on sharpness of threshold.

The result has been extended to hypergraphs:
Dyer, Frieze and Greenhill (2014).

Achlioptas and Naor showed that for approximately half of the possible values for d, $\chi(G_{n,d/n})$ is determined w.hp.

Theorem (Achlioptas and Naor (2005))

If $d \in ((2k-1)\log k, 2k\log k)$ *then w.h.p.* $\chi(G_{n,d/n}) = k+1$.

This has been improved so that we now have

Theorem (Coja-Oghlan and Vilenchik (2013))

(a) Let κ_d *be the smallest integer* $k \geq 2$ *such that* $d < (2k-1)\log k$. *Then* $\chi(G_{n,d/n}) = \kappa_d$ *for* $d \in \mathcal{A}$ *where* $\mathcal{A}$ *has asymptotic density one in* R_+.

(b) $d_k > 2k\log k - \log k - 2\log 2 + o_k(1)$.

Chromatic Number

Upper bound on d_k: Let $X_k(d)$ denote the number of k-colorings of $G_{n,d/n}$. Then

$$d > 2k \log k - \log k \text{ implies } \mathbf{E}(X_k(d)) \to 0$$

and therefore

$$d_k < 2k \log k - \log k.$$

Chromatic Number

Upper bound on d_k:

$$d_k < 2k \log k - \log k.$$

Theorem (Coja-Oghlan (2014))

$$d_k \le 2k \log k - \log k - 1 + o_k(1).$$

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This problem has attracted the attention of Statistical Physicists where colors are synonyms for spins. Coja-Oghlan's proof is motivated by physicists conjectures about the geometry of the set of k-colorings near the threshold. His upper bound matches a physics prediction.

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For large k, the value of d_k is now known within an interval of size less than 0.39.

Contents of talk

(a) Random Discrete Structures

(b) Random Instances of the TSP in the unit square $[0,1]^2$

(c) The Random Graphs $G_{n,m}$ and $G_{n,p}$.

  (1) Evolution
  (2) Chromatic number
  (3) Matchings
  (4) Hamilton cycles

(d) Randomly edge weighted graphs

  1. Minimum Spanning Tree
  2. Shortest Paths
  3. 3-Dimensional Assignment Problem
  4. Random Instances of the TSP with independent costs

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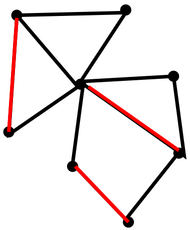
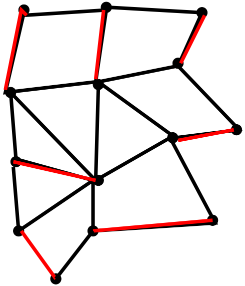
(f) Open Problems

Matchings

A matching in a graph G is a set of vertex disjoint edges. The matching is perfect if every vertex is covered by an edge of the matching.



Matching



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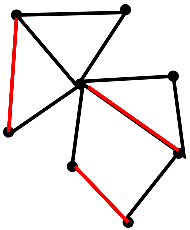

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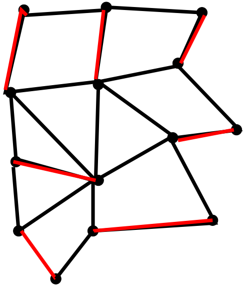


Perfect Matching

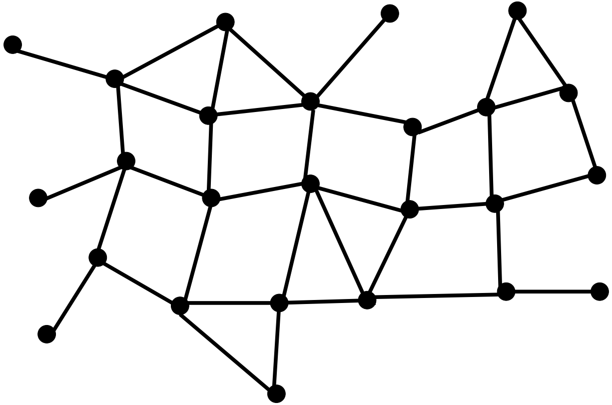
A largest matching can be found in polynomial time Edmonds (1965).

Karp and Sipser (1981) proposed the following greedy algorithm for finding a large matching:

KSGREEDY

begin
$M \leftarrow \emptyset$;
while $E(G) \neq \emptyset$ **do**
begin
A1: If G has a vertex of degree one, choose one, x say, randomly.
Let $e = \{x, y\}$ be the unique edge of G incident with x;
A2: Otherwise, (no vertices of degree one) choose $e = \{x, y\} \in E$ randomly
$G \leftarrow G \setminus \{x, y\}$;
$M \leftarrow M \cup \{e\}$
end;
Output M
end

Matchings



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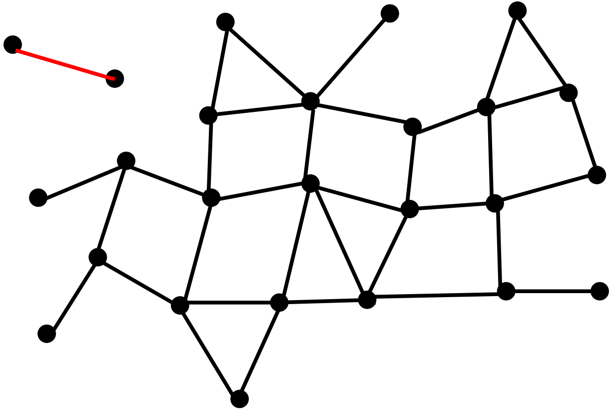

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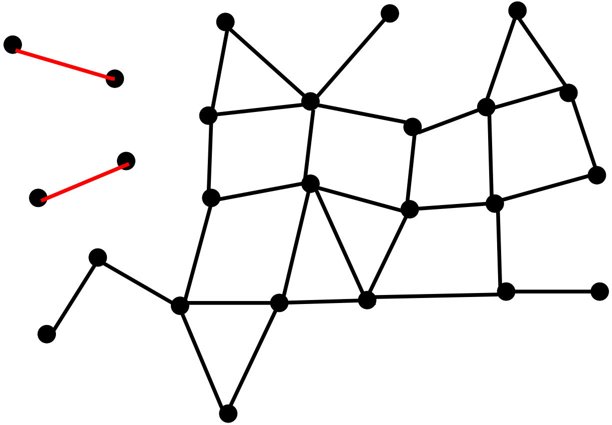

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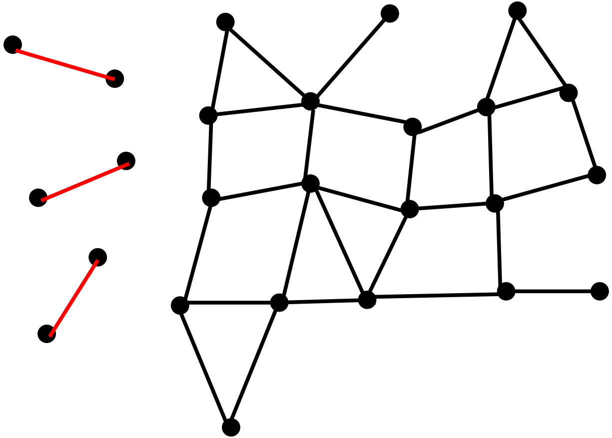

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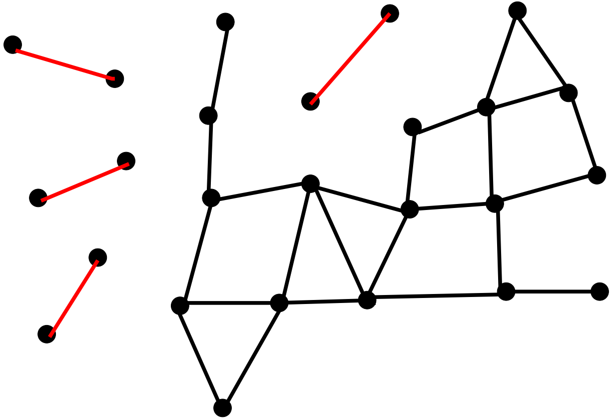

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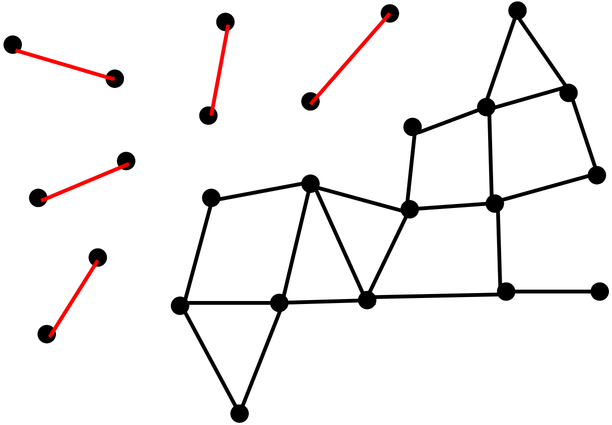

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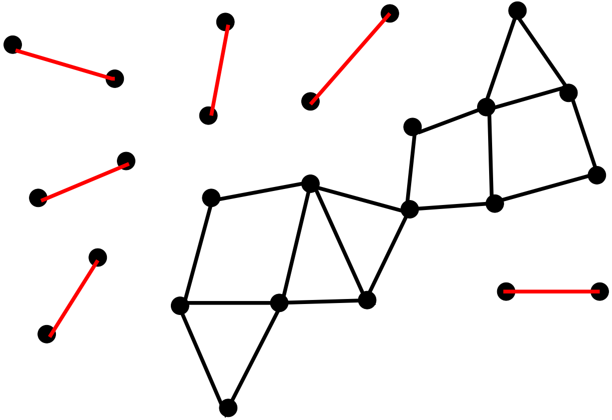
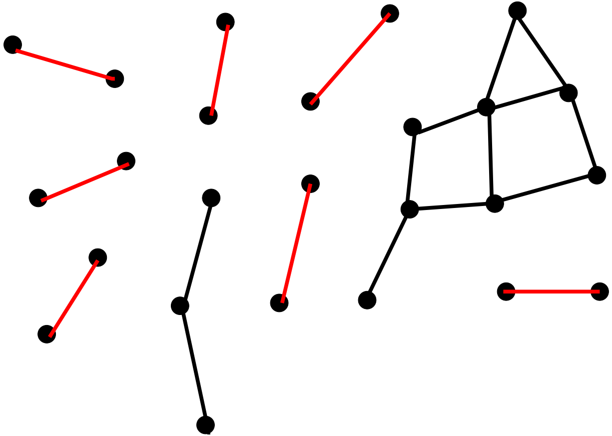


End of Phase 1

Matchings



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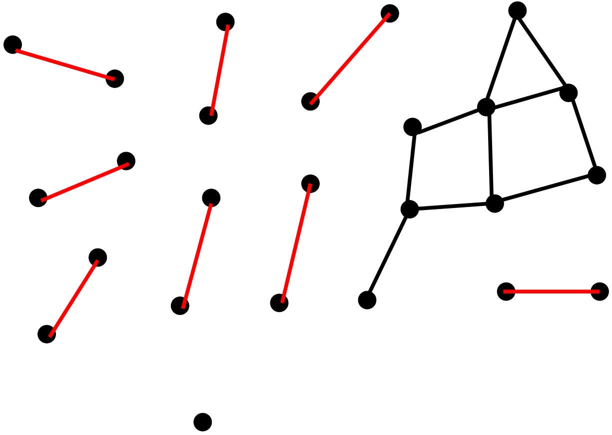

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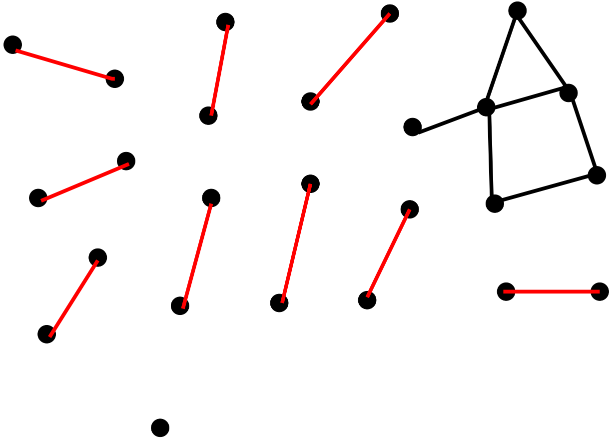

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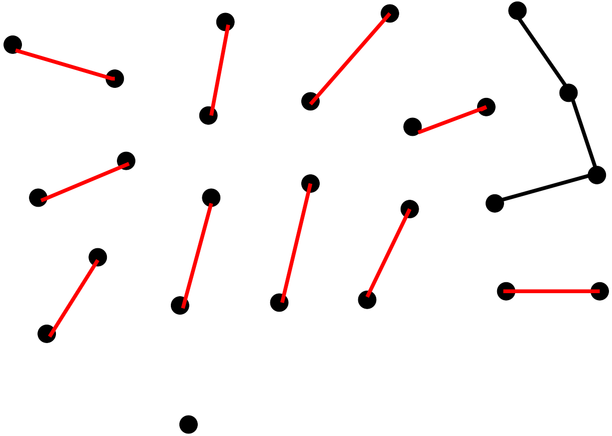

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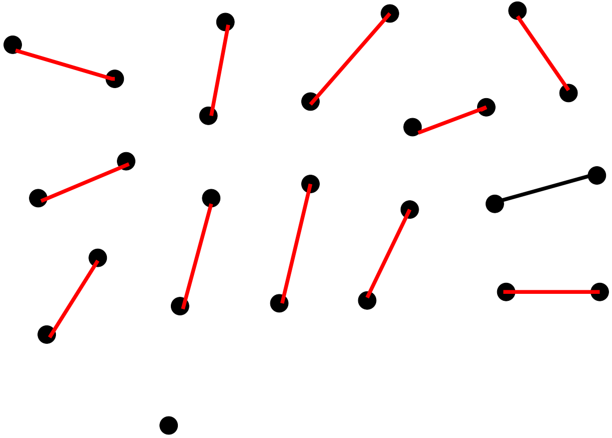

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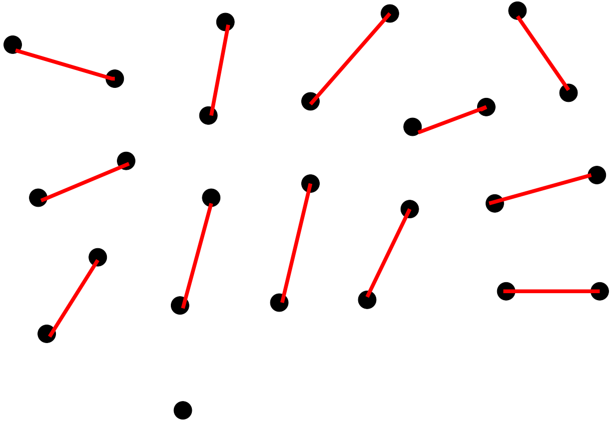

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In $G_{n,p}$ each of the $\binom{n}{2}$ edges of the complete graph K_n appear independently with probability p.
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If $c \geq e$ then w.h.p. Phase 2 isolates $o(n)$ vertices.
If $c \geq e$ then w.h.p. Phase 2 isolates $\Theta(n^{1/5} \log^{O(1)} n)$ vertices.

Matchings

For the graph G remaining after t steps of the algorithm, let

$$\begin{aligned} v_1 &= \text{the number of vertices of degree one} \\ v &= \text{the number of vertices of degree at least two} \\ m &= \text{the number of edges} \end{aligned}$$

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- At each stage G is a random graph with these parameters.
- The number of vertices v_k of degree $k \geq 2$ satisfies

$$v_k \approx \frac{vz^k}{k!(e^z - 1 - z)}$$

where z is the solution to

$$\frac{2m - v_1}{v} = \frac{z(e^z - 1)}{e^z - 1 - z}$$

One step transitions:

If v_1', v', m' denote the values of the parameters after one step of the algorithm then, given v_1, v, m

$$\mathbf{E}[v_1' - v_1] = -1 - \frac{v_1}{2m} + \frac{v^2 z^4 e^z}{(2mf)^2} - \frac{v_1 v z^2 e^z}{(2m)^2 f} + O\left(\frac{\log^2 v}{vz}\right),$$

$$\mathbf{E}[v' - v] = -1 + \frac{v_1}{2m} - \frac{v^2 z^4 e^z}{(2mf)^2} + O\left(\frac{\log^2 v}{vz}\right),$$

$$\mathbf{E}[m' - m] = -1 - \frac{vz^2 e^z}{2mf} + O\left(\frac{\log^2 v}{vz}\right).$$

$$\mathbf{E}[v_0' - v_0] = O\left(\frac{v_1}{m}\right)$$ — expected increase in unmatched vertices.

One step transitions:

If v_1', v', m' denote the values of the parameters after one step of the algorithm then, given v_1, v, m

$$\mathbf{E}[v_1' - v_1] = -1 - \frac{v_1}{2m} + \frac{v^2 z^4 e^z}{(2mf)^2} - \frac{v_1 v z^2 e^z}{(2m)^2 f} + O\left(\frac{\log^2 v}{vz}\right),$$

$$\mathbf{E}[v' - v] = -1 + \frac{v_1}{2m} - \frac{v^2 z^4 e^z}{(2mf)^2} + O\left(\frac{\log^2 v}{vz}\right),$$

$$\mathbf{E}[m' - m] = -1 - \frac{vz^2 e^z}{2mf} + O\left(\frac{\log^2 v}{vz}\right).$$

$$\mathbf{E}[v_0' - v_0] = O\left(\frac{v_1}{m}\right)$$ — expected increase in unmatched vertices.

v_1, v, m closely follow the trajectory of a set of differential equations.

These equations are:

$$\frac{dv_1}{dt} = -1 - \frac{v_1}{2m} + \frac{v^2 z^4 e^z}{(2mf)^2} - \frac{v_1 v z^2 e^z}{(2m)^2 f},$$

$$\frac{dv}{dt} = -1 + \frac{v_1}{2m} - \frac{v^2 z^4 e^z}{(2mf)^2},$$

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Their solution is

$$2m = \frac{n}{c} z^2,$$
$$v = n(1 - e^{-z}(1+z))\beta,$$
$$v_1 = \frac{n}{c}\left[z^2 - zc\beta(1 - e^{-z})\right],$$
$$t = \frac{n}{c}\left[c(1-\beta) - \frac{1}{2}\log^2\beta\right],$$

where $\beta e^{c\beta} = e^z$.

Super-critical case: $c > e$

In this case we end Phase 1 with $z = z^* > 0$.

Super-critical case: $c > e$

In this case we end Phase 1 with $z = z^* > 0$.

We have observed that

$\mathbf{E}[v_0' - v_0] = O\left(\frac{v_1}{m}\right)$ — expected increase in unmatched vertices.

So, it is enough to show that w.h.p. $v_1 = \tilde{O}(n^{1/5})$ throughout the algorithm, for then we can argue that w.h.p. there are $\tilde{O}\left(n^{1/5}\sum_{m=1}^{cn}\frac{1}{m}\right)$ vertices left unmatched in Phase 2.

Controlling v_1

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We first observe that

$$v_1 > 0 \text{ implies } \mathbf{E}[v_1' - v_1] \leq -\min\left(\frac{z^2}{200}, \frac{1}{20000}\right) + O\left(\frac{\log^2 n}{vz}\right)$$

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Early Phase: $z \geq n^{-1/100}$.
Whp v_1 stays $\tilde{O}(z^{-2})$.

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Middle Phase: $n^{-1/100} \geq z \geq n^{-1/5}$
At this point the graph is very sparse, most vertices are of degree two.
When $v_1 > 0$ most vertices of degree one are at end of a long path. Removing such a vertex and its edge does not change v_1 i.e.

$$\mathbf{Pr}(v_1' = v_1 \mid v_1 > 0) = 1 - z + O(z^2).$$

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Final Phase: $z \leq n^{-1/5}$
We start this phase with

$$v \sim v_2 \sim Cnz^2 = \tilde{O}(n^{3/5})$$

Only $\tilde{O}(n^{3/5}z) = \tilde{O}(n^{2/5})$ moves made in the "v_1 walk" and so v_1 can only move by square root of this.

Matchings

The Karp-Sipser algorithm runs in $O(n)$ time and makes $\tilde{O}(n^{1/5})$ mistakes.

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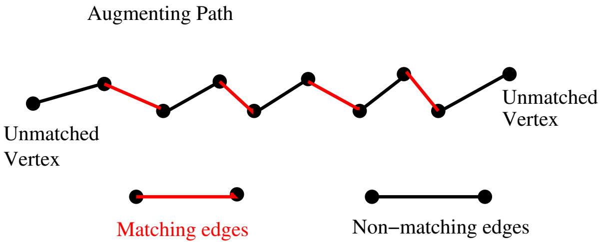
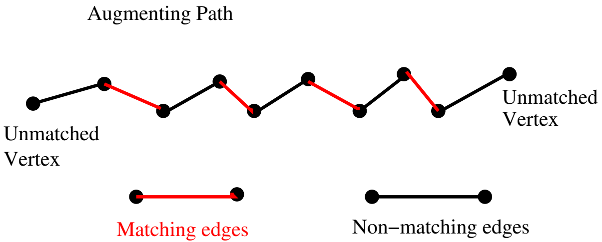
All statements from now on refer to Phase 2.

Matchings



Replacing the matching edges by non-matching edges on the path, and only the path, yields a larger matching.



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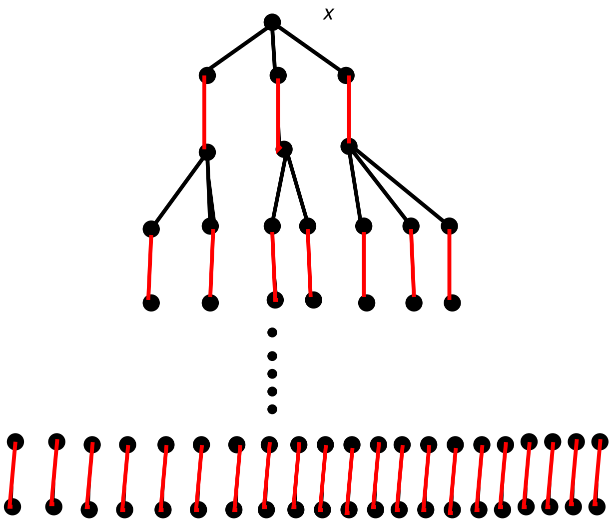
To find an augmenting path from unmatched vertex x to vertex unmatched vertex y, we use augmenting trees:

Matchings



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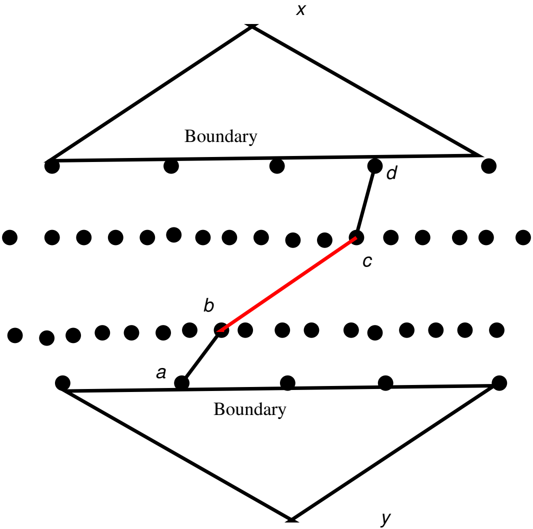

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When a regular vertex is deleted, it will be matched to the first available edge in the order. The next edge in the order containing v is called the witness for v.

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$$R_0 = \{v \in R : \ v \text{ is early and the witness of } v \text{ is punctual}\}.$$

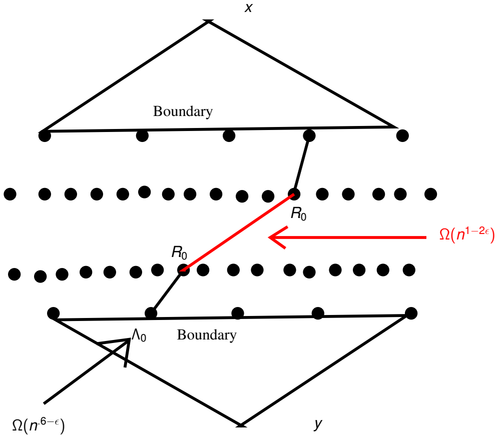
and

$$\Lambda_0 = \left\{ v : v \text{ has punctual degree at least ten in } G(n^{1-\epsilon}) \right\}$$

where $G(t)$ is the graph G after t steps of Phase 2.

The tardy $R_0 : \Lambda_0$ edges are uniformly random from $R_0 \times \Lambda_0$, conditional on all other edges.
This is because they do not affect the course of the algorithm.

These values show an expected $\Omega(n^{.2-4\epsilon})$ paths.
As such, w.h.p., we succeed in finding augmenting paths.



Contents of talk

(a) Random Discrete Structures

(b) Random Instances of the TSP in the unit square $[0, 1]^2$

(c) The Random Graphs $G_{n,m}$ and $G_{n,p}$.

  (1) Evolution
  (2) Chromatic number
  (3) Matchings
  (4) Hamilton cycles

(d) Randomly edge weighted graphs

  1. Minimum Spanning Tree
  2. Shortest Paths
  3. 3-Dimensional Assignment Problem
  4. Random Instances of the TSP with independent costs

(e) Random k-SAT

(f) Open Problems

Hamilton Cycles

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Theorem (Komlós and Szemerédi (1983))

Suppose that $m = \frac{1}{2}n(\log n + \log\log n + c_n)$. *Then*

$$\lim_{n\to\infty} \mathbf{Pr}(G_{n,m} \text{ is Hamiltonian}) = \begin{cases} 0 & c_n \to -\infty \\ e^{-e^{-c}} & c_n \to c \\ 1 & c_n \to \infty \end{cases}$$

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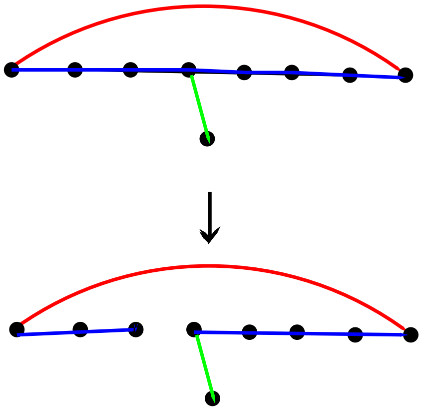
We will describe an algorithm that runs in polynomial time and finds a Hamilton cycle w.h.p. for the case $c_n = \omega \to \infty$.

Posá Rotations

We can start our algorithm with any path.

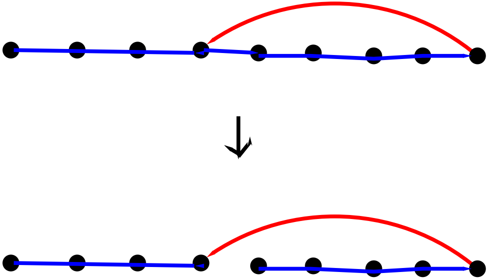
The red edge extends the blue path.

Alternative way of extending path:



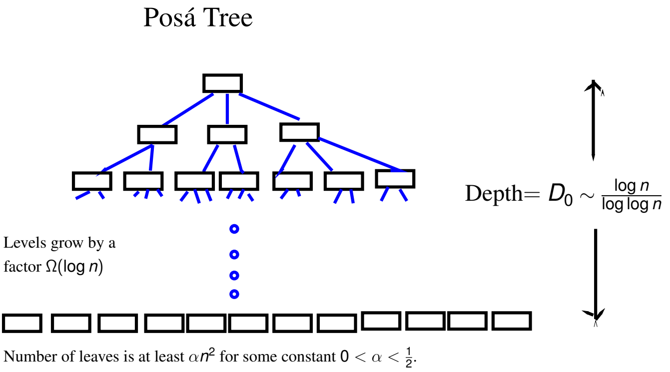
Hamilton Cycles

If there is no extension then we rotate the path:



We will in general, have several choices for the red edge here.
Each rotation gives another endpoint.

Hamilton Cycles



Each rectangle is a path that is obtained from its parent by a rotation.

BOOST is the set of pairs of endpoints in the leaves.

Hamilton Cycles

Let $m = \frac{1}{2}n(\log n + \log\log n + \omega)$ and $m_2 = \omega n/4$ and let $m_1 = m - m_2$.

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It is not necessary to partition the edges and the algorithm can be made deterministic, Bollobás, Fenner and Frieze (1985).

Hamilton cycles

With the threshold problem solved, existentially and constructively, we can consider other models of a random graph: We first see what happens if we condition on minimum degree at least two:

Hamilton cycles

Let $G_{n,m;k}$ be sampled uniformly from all graphs with vertex set $[n]$ that have m edges and minimum degree at least k.

Theorem (Bollobás, Fenner and Frieze (1990))

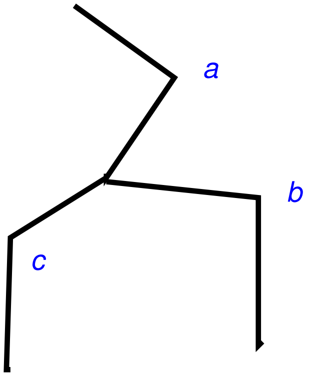
Let $m = \frac{1}{6}n(\log n + \log\log n + c_n)$ *then*

$$\lim_{n\to\infty} \mathbf{Pr}(G_{n,m;2} \text{ is Hamiltonian}) = \begin{cases} 0 & c_n \to -\infty \\ e^{-f(c)} & c_n \to c \\ 1 & c_n \to +\infty \end{cases}$$

for some explicit function $f(c)$*.*

$e^{-f(c)}$ is the asymptotic probability that there are no spiders.

Spiders



Vertices a, b, c are of degree two

Hamilton cycles

Let $G(n,r)$ denote a random r-regular graph chosen uniformly from the set of all graphs with vertex set $[n]$.
(Regular means that all vertices have the same degree)

Theorem

$$\lim_{n\to\infty} \mathbf{Pr}(G(n,r) \text{ is Hamiltonian}) = 1, \quad r \geq 3.$$

$r = O(1)$ was proved by Robinson and Wormald (1992,1994)
$r \to \infty$ was proved by Krivelevich, Sudakov, Vu, Wormald (2001) and Cooper, Frieze, Reed (2002).

If each vertex independently chooses k random neighbors then we have the random graph G_{k-out}.

Theorem (Bohman and Frieze (2009))

$$\lim_{n\to\infty} \mathbf{Pr}(G_{k-out} \text{ is Hamiltonian}) = 1, \quad k \geq 3.$$

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This is not implied by the previous results on random regular graphs.
We need $k \geq 3$ to avoid spiders:

We now consider conditoning on minimum degree at least three. Let

$$L_c = \lim_{n\to\infty} \mathbf{Pr}(G_{n,cn;3} \text{ is Hamiltonian})$$

Conjecture: $L_c = 1$ for all $c \geq 3/2$.

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Theorem (Bollobás, Cooper, Fenner, Frieze (2000))

$L_c = 1$ *for* $c \geq 128$.

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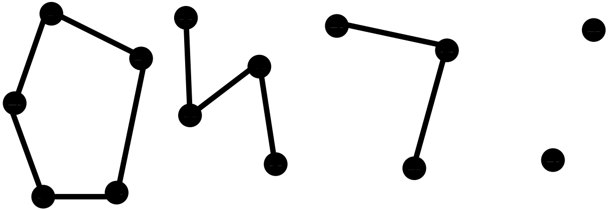
Conjecture true for $c \geq 3$. Assuming numerical solution of some differential equations.

Theorem (Frieze and Haber (2014))

If c is sufficiently large then w.h.p. a Hamilton cycle can be found in $G_{n,cn;3}$ in $O(n^{1+o(1)})$ time.

The improved results on Hamilton cycles in $G_{n,cn;3}$ rely on the analysis of a greedy algorithm for finding a good 2-matching M viz. a set of edges that induce a graph of maximum degree at most two.



By good, we mean that M has $O(\log n)$ components. This gives us a good basis for constructing a Hamilton cycle.
We next discuss an algorithm for finding such a 2-matching.

Algorithm 2GREEDY: The input for this algorithm is $G_{n,cn}^{\delta\geq 3}$ for c suficiently large – currently $c \geq 10$ will suffice.

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Otherwise we choose a random edge incident to a vertex in $\bar{B}$.
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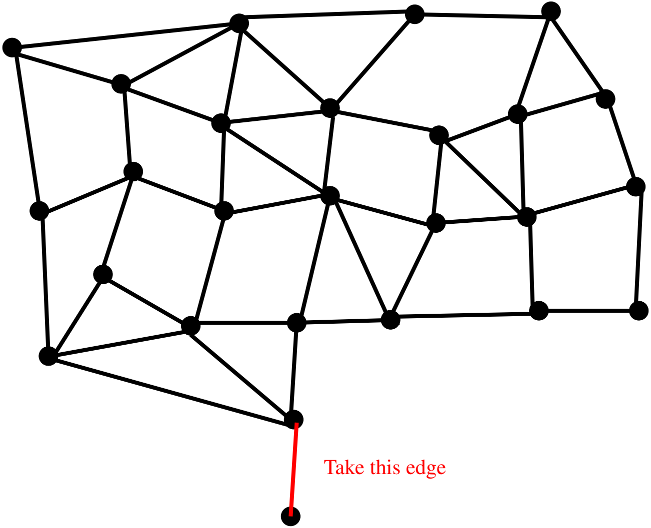
Phase 1 ends when $\bar{B} = \emptyset$.

Hamilton cycles



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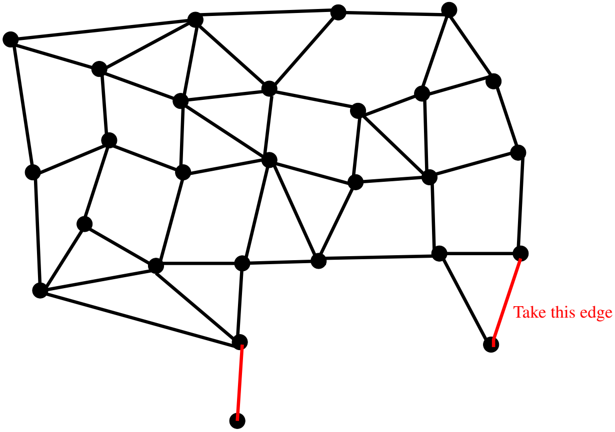

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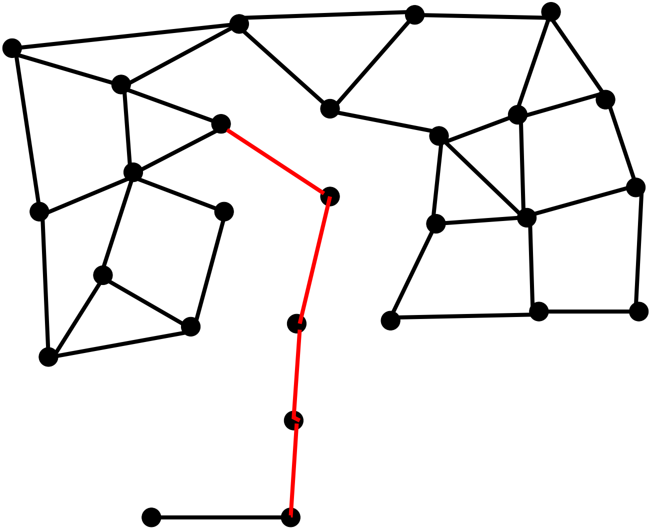
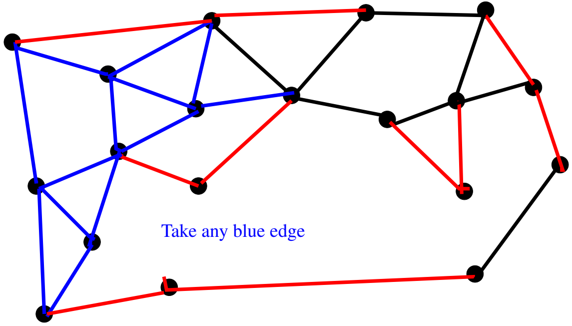


Take this edge

If none of these cases are applicable then we choose a random edge among those incident with a vertex not in *B* i.e. not yet covered by *M*.



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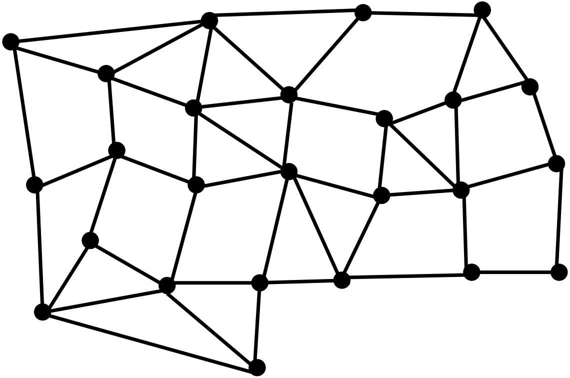

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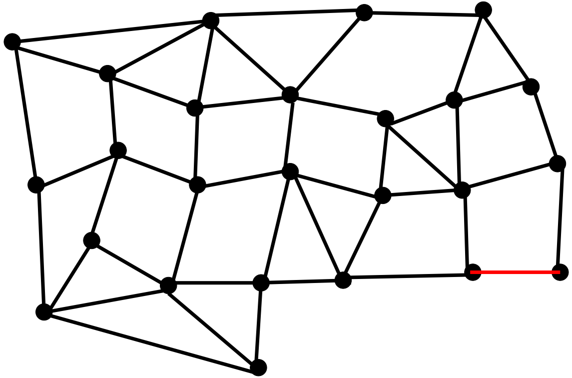

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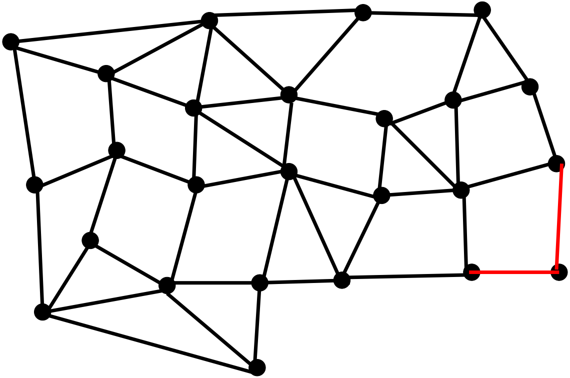

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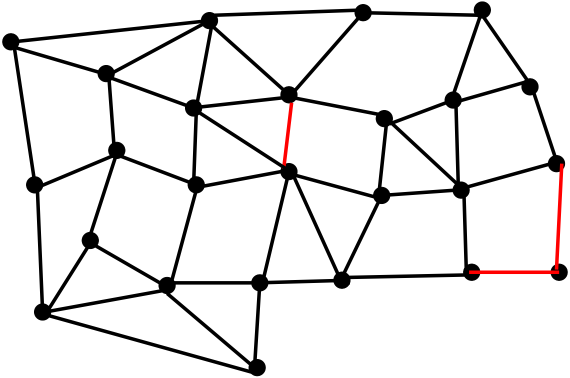

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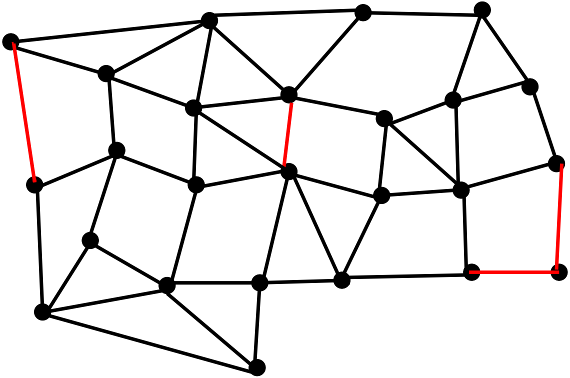

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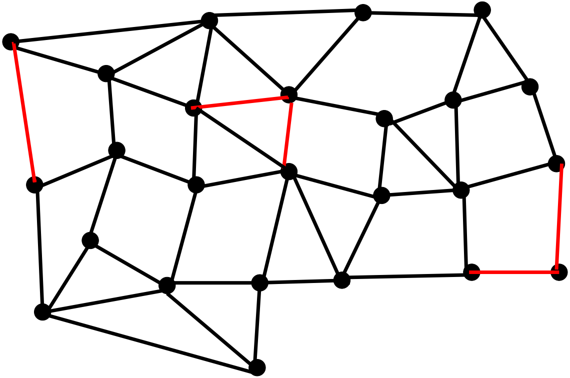

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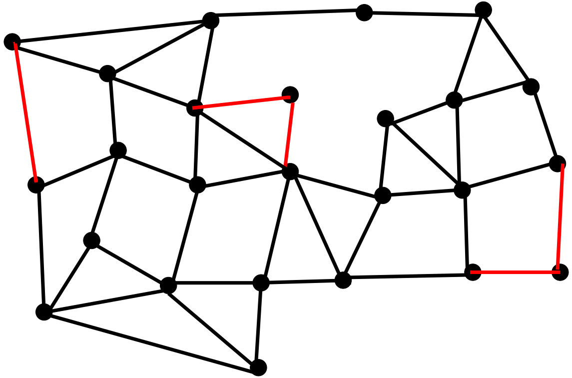

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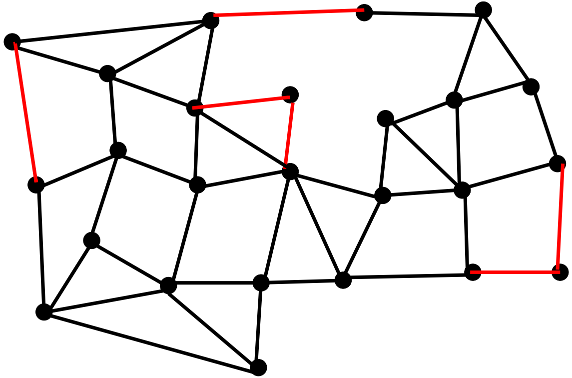

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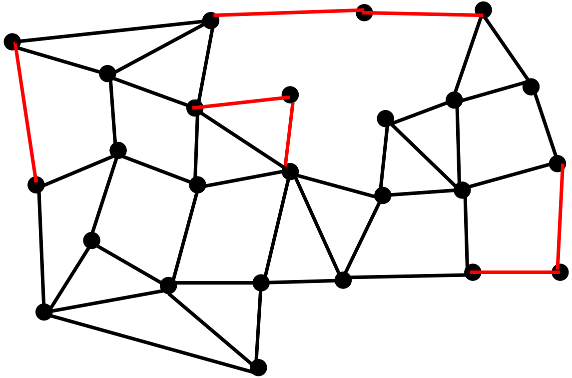

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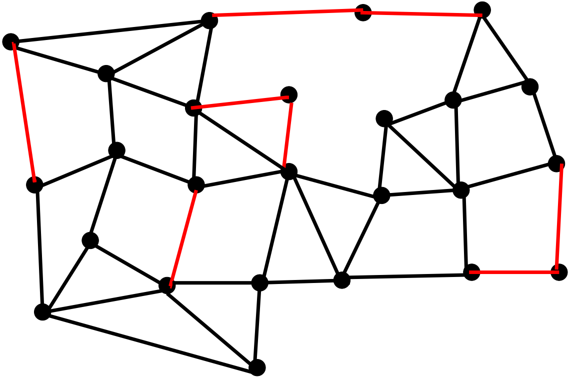

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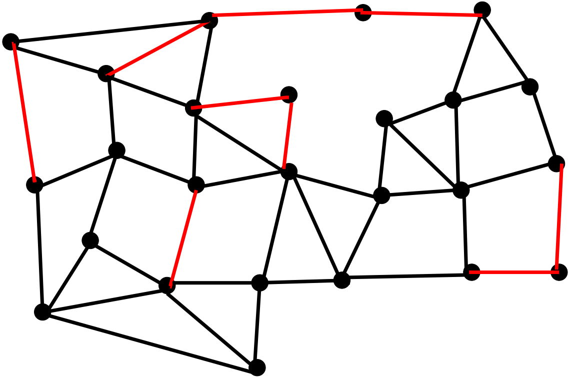

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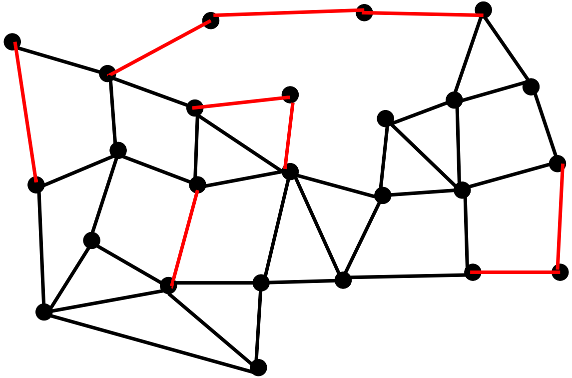

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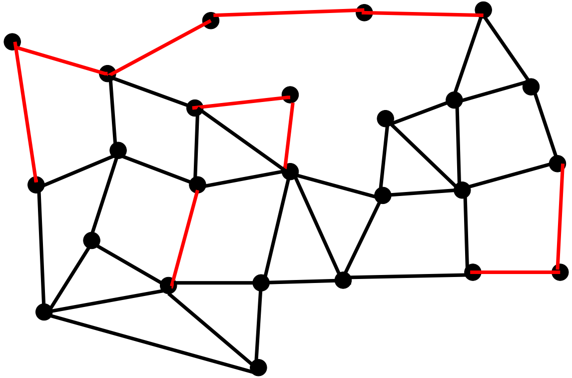

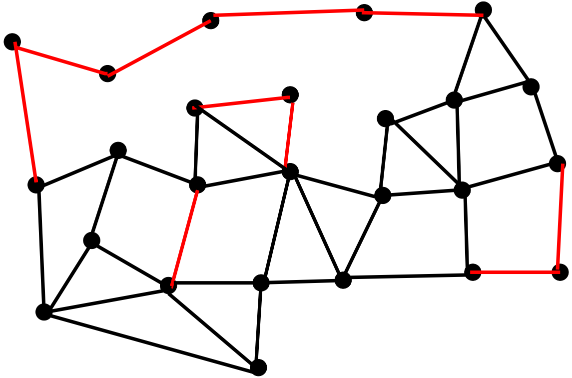

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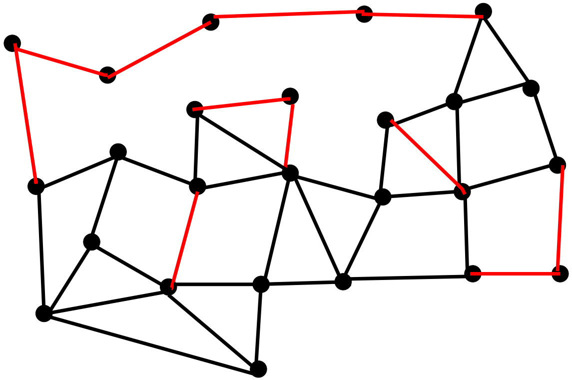

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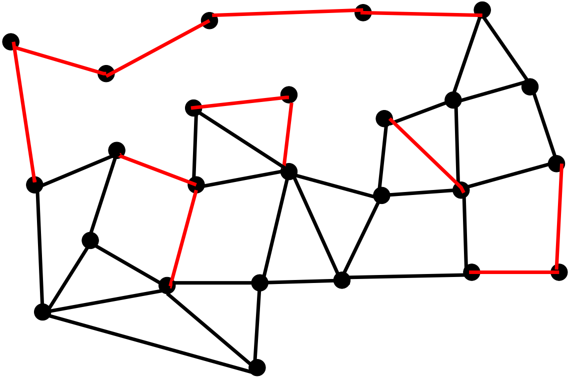

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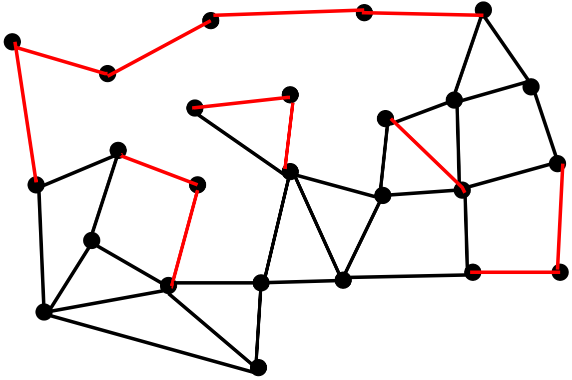

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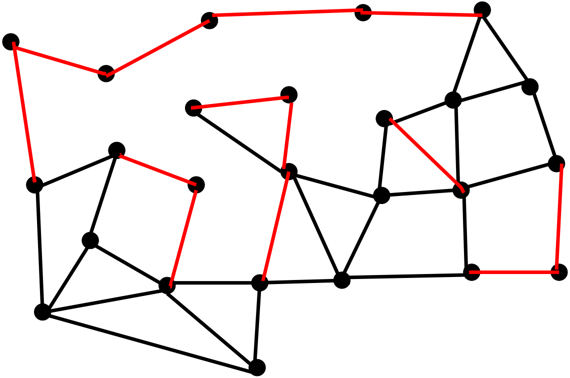

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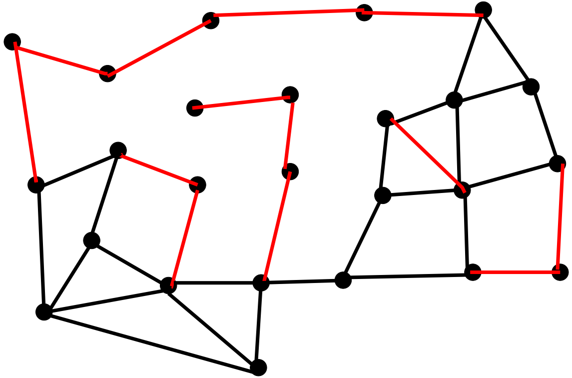

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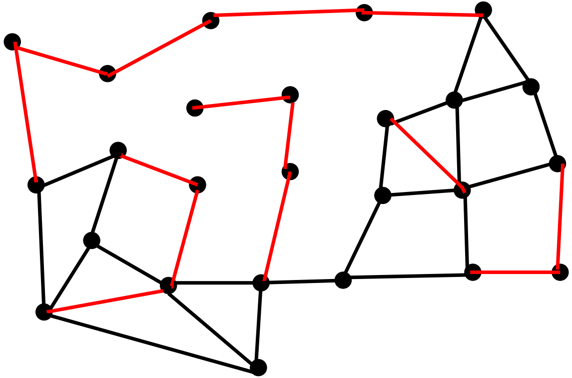

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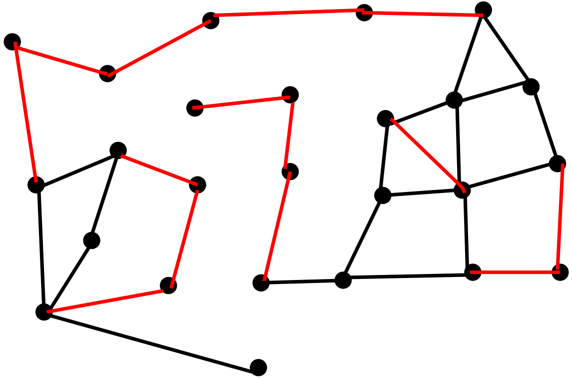

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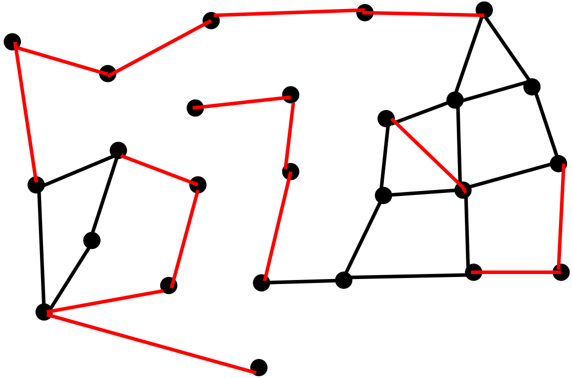

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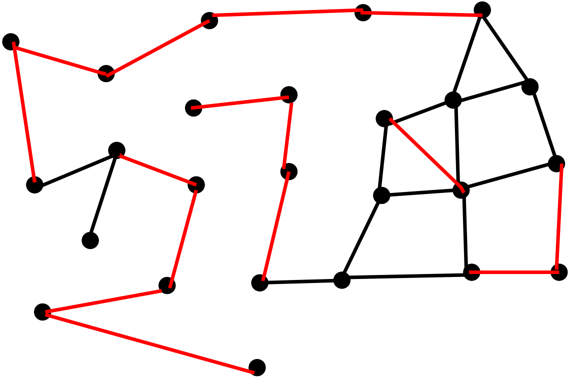

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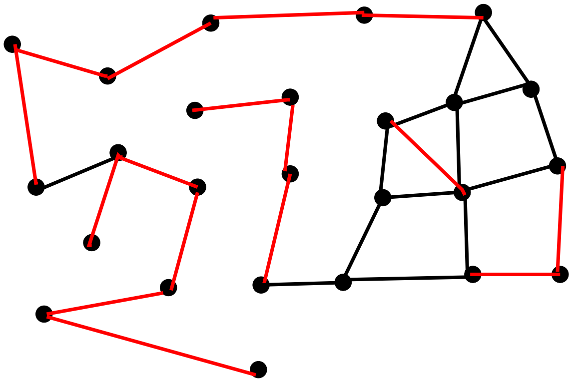

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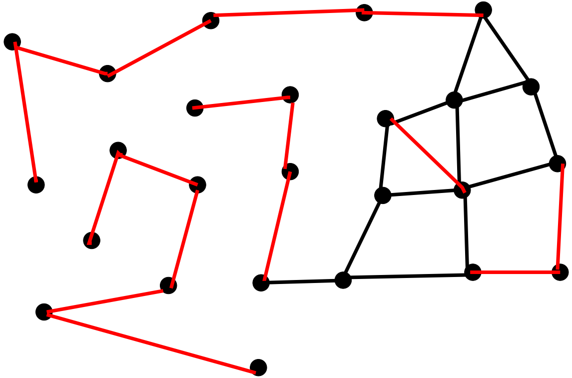

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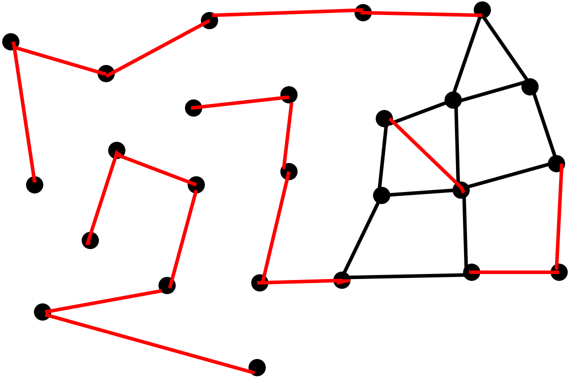

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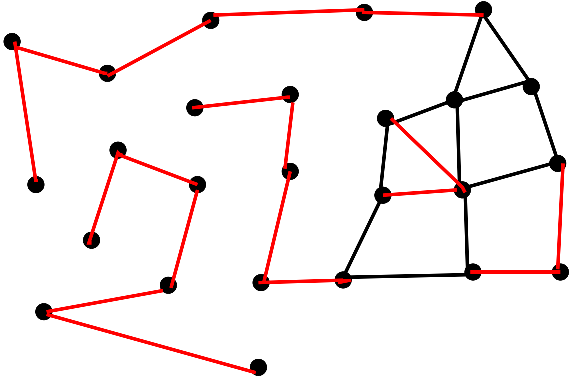

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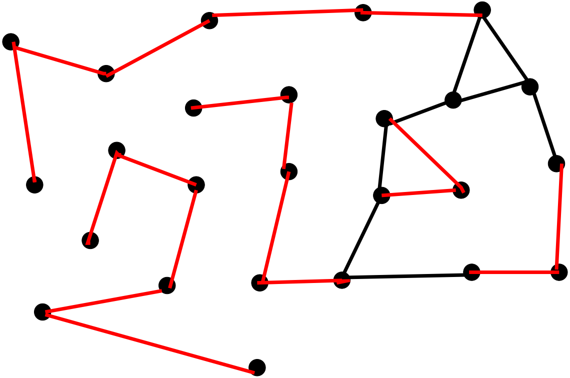

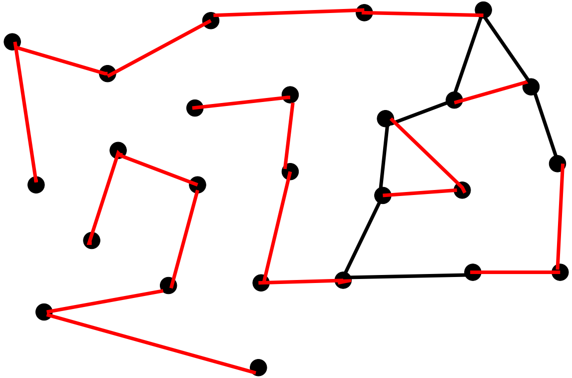

Phase 1 is over.

At the end of Phase 1, the 2-matching M will consist mainly of vertex disjoint paths. The isolated vertices and the cycles will play no further part in the rest of the 2GREEDY algorithm. They will be part of the output though.

We show that w.h.p. the number of paths is $\Omega(n)$ and that there are $O(\log n)$ isolated vertices.

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We find a perfect matching M' in H in $O(n)$ time. Adding M to M' produces a 2-matching in G which has $O(\log n)$ components w.h.p.

The analysis of 2-GREEDY is similar to that of the Karp-Sipser algorithm: Only, it has more parameters:

- μ is the number of edges in Γ,
- $y_k = |Y_k| = |\{v \notin B : d_\Gamma(v) = k\}|$, $k = 1, 2$,
- $z_1 = |Z_1| = |\{v \in B : d_\Gamma(v) = 1\}|$,
- $y = |Y| = |\{v \notin B : d_\Gamma(v) \geq 3\}|$.
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We can show that y_1, y_2, z_1 remain $O(\log n)$ throughout, w.h.p. And that Phase 1 ends with $y = 0$ and $z_1 = \Omega(n)$.

As 2-GREEDY progresses the random sequence $(y_0, y_1, y_2, z_1, y, z, \mu)$ is a Markov Chain.

The graph defined by the remaining vertices and edges is chosen uniformly from the set of graphs with these parameters.

The degrees of vertices in Y, Z are close to truncated Poisson:

Let $f_i(x) = e^x - \sum_{t=0}^{i-1} \frac{x^t}{t!}$ and let λ be the solution to

$$\frac{y\lambda f_2(\lambda)}{f_3(\lambda)} + \frac{z\lambda f_1(\lambda)}{f_2(\lambda)} = 2\mu - y_1 - 2y_2 - z_1.$$

Then w.h.p.

$$y_k \sim \frac{y\lambda^k}{k! f_3(\lambda)}, k \geq 3 \text{ and } z_k \sim \frac{z\lambda^k}{k! f_2(\lambda)}, k \geq 2.$$

There are differential equations that closely model the process: They only involve variables $\hat{y}, \hat{z}, \hat{\mu}$ that represent y, z, μ, other variables stay small.
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$$\frac{d\hat{y}}{dt} = \hat{A} + \hat{B} - \hat{C} - 1; \quad \frac{d\hat{z}}{dt} = 2\hat{C} - 2\hat{A} - 2\hat{B}; \quad \frac{d\hat{\mu}}{dt} = -1 - \hat{D}$$

where

$$\hat{A} = \frac{\hat{y}\hat{z}\hat{\lambda}^5 f_0(\hat{\lambda})}{8\hat{\mu}^2 f_2(\hat{\lambda}) f_3(\hat{\lambda})}, \ \hat{B} = \frac{\hat{z}^2\hat{\lambda}^4 f_0(\hat{\lambda})}{4\hat{\mu}^2 f_2(\hat{\lambda})^2}, \ \hat{C} = \frac{\hat{y}\hat{\lambda} f_2(\hat{\lambda})}{2\hat{\mu} f_3(\hat{\lambda})}, \ \hat{D} = \frac{\hat{z}\hat{\lambda}^2 f_0(\hat{\lambda})}{2\hat{\mu} f_2(\hat{\lambda})}$$

and

$$\frac{\hat{y}\hat{\lambda} f_2(\hat{\lambda})}{f_3(\hat{\lambda})} + \frac{\hat{z}\hat{\lambda} f_1(\hat{\lambda})}{f_2(\hat{\lambda})} = 2\hat{\mu}.$$

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Unfortunately, we have not been able to solve these equations.

We observe however, that if $\hat{\lambda}$ is large then

$$\hat{A} \ll 1; \quad \hat{B} \ll 1; \quad \hat{C} \approx \frac{\hat{y}\hat{\lambda}}{2\hat{\mu}}; \quad \hat{D} \approx \frac{\hat{z}\hat{\lambda}^2}{2\hat{\mu}}; \quad \hat{\lambda} \approx \frac{2\hat{\mu}}{\hat{y}+\hat{z}}.$$

They can then be approximated by the following equations:

$$\begin{aligned}
\tilde{y}' &= -\frac{\tilde{y}}{\tilde{y}+\tilde{z}} - 1\\
\tilde{z}' &= \frac{2\tilde{y}}{\tilde{y}+\tilde{z}}\\
\tilde{\mu}' &= -1 - \frac{2\tilde{z}\tilde{\mu}}{(\tilde{y}+\tilde{z})^2}\\
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And then because the diferential equations describe the process very closely, we can deduce that w.h.p. there is a time T such that $y_1(T) = y_2(T) = z_1(T) = \zeta(T) = y(T) = 0$ and $z(T) = \Omega(n)$.

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In which case, as already noted, we will end Phase 1 having only isolated $O(\log n)$ vertices.

Hamilton cycles

Numerical experiments suggest that such a T exists for $c \geq 2.5$, maybe even for smaller c.

c	y_{final}	z_{final}	μ_{final}	λ_{final}
3.0	0.000008	0.283721	0.398527	1.822428
2.9	0.000009	0.242563	0.326139	1.602749
2.8	0.000010	0.197461	0.253645	1.370798
2.7	0.000010	0.148901	0.182327	1.123928
2.6	0.000010	0.098344	0.114494	0.858355
2.5	0.000010	0.048976	0.054010	0.565840

These are the results of running Euler's method with step length 10^{-5} on the differential equations.

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We define R_0, Λ_0 more or less as before

$$R_0 = \{v \in R : \; v \text{ is early and the witness of } v \text{ is punctual}\}.$$
$$\Lambda_0 = \left\{v : v \text{ has punctual degree at least ten in } G(n^{1-o(1)})\right\}$$

Once again, the tardy $R_0 : \Lambda_0$ edges are uniformly random from $R_0 \times \Lambda_0$, conditional on all other edges.

Hamilton cycles

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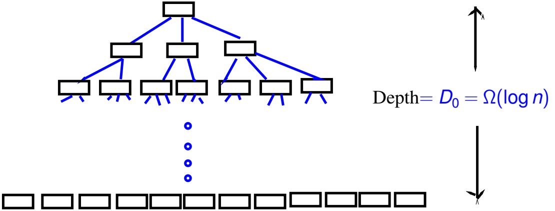
We try to grow our path using extensions and rotations. With a given path P with endpoints v, w we grow a Posa tree with v as one endpint of all the paths produced.

Posá Tree: rotations with one endpoint fixed.



Number of leaves in Λ_0 is at least $n^{1/2+o(1)}$.

In the above diagram, each rectangle is a path that is obtained

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if we fail to extend, then the probability we fail to find a tardy $R_0 : \Lambda_0$ edge joining $x \in END$ to $y \in END(x)$ is $n^{-o(1)}$.

We only need to extend/close a cycle $O(\log^2 n)$ times and so the probability we fail is $O(n^{-o(1)} \log^2 n) = o(1)$, if we are careful with our $o(1)$'s.

So, for c sufficiently large we can find a Hamilton cycle in $G_{n,cn}^{\delta \geq 3}$ in $O(n^{1+o(1)})$ time.

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(b) Random Instances of the TSP in the unit square $[0,1]^2$

(c) The Random Graphs $G_{n,m}$ and $G_{n,p}$.

  (1) Evolution
  (2) Chromatic number
  (3) Matchings
  (4) Hamilton cycles

(d) Randomly edge weighted graphs

  1. Minimum Spanning Tree
  2. Shortest Paths
  3. 3-Dimensional Assignment Problem
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(e) Random k-SAT

(f) Open Problems

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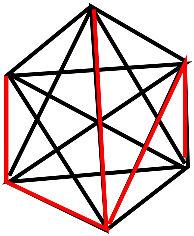
Minimum Length Spanning Tree

Every edge e of the complete graph K_n is given a random length X_e.

The edge lengths are independently uniform $[0, 1]$ distributed.

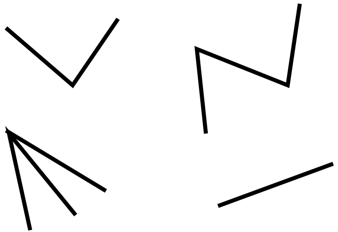
Z_n is the minimum total length of a spanning tree i.e. a connected subgraph that contains $n - 1$ edges and no cycles.



Spanning Tree

Length=sum of lengths of edges.

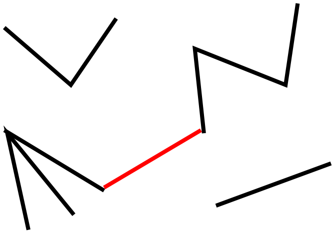
Greedy Algorithm



F is the forest induced by the edges chosen so far.

F has 4 components.

Greedy Algorithm



F is the forest induced
by the edges chosen so far.

Edges between components are longer
than edges inside components

The algorithm adds the shortest edge joining components of F.

The algorithm adds longer and longer edges as it progresses.

Minimum Length Spanning Tree

If ℓ_F is the length of the longest edge in F, then edges of length at most ℓ are contained in the components of F.

Therefore the algorithm adds $\kappa - 1$ more edges, where κ is the number of components in the graph spanned by edges of length at most ℓ_F.

Minimum Length Spanning Tree

Let T be the minimum spanning tree and let ℓ denote length.

$$\begin{aligned} Z_n = \ell(T) &= \sum_{e \in T} X_e \\ &= \sum_{e \in T} \int_{p=0}^{1} 1_{(p \le X_e)} dp \\ &= \int_{p=0}^{1} \sum_{e \in T} 1_{(p \le X_e)} dp \\ &= \int_{p=0}^{1} |\{e \in T : p \le X_e\}| dp \end{aligned}$$

$$\ell(T) = \int_{p=0}^{1} |\{e \in T : X_e \geq p\}| dp$$
$$= \int_{p=0}^{1} (\kappa(G_p) - 1) dp,$$

where $\kappa(G_p)$ is the number of components in the graph induced by edges of length at most p.

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where $\kappa(G_p)$ is the number of components in the graph induced by edges of length at most p.

So

$$\mathbf{E}(Z_n) = \int_{p=0}^{1} (\mathbf{E}(\# \text{ components in } G_{n,p}) - 1) dp.$$

Minimum Length Spanning Tree

$$\mathbf{E}(Z_n) = \int_{p=0}^{1} (\mathbf{E}(\#\text{ components in } G_{n,p}) - 1)dp.$$

FACT: $p \geq 6 \log n/n$ implies that $G_{n,p}$ is connected with sufficiently high probability.

FACT: Almost all of the integral is accounted for by small isolated tree components.

So,

$$\mathbf{E}(Z_n) \sim \int_{p=0}^{6 \log n/n} \mathbf{E}(\#\text{ small isolated trees in } G_{n,p})dp.$$

Minimum Length Spanning Tree

$$\mathbf{E}(Z_n) \sim \int_{p=0}^{6\log n/n} \mathbf{E}(\#\text{ small isolated trees in } G_{n,p})dp$$

$$\sim \int_{p=0}^{6\log n/n} \left(\sum_{k=1}^{\log^2 n} \binom{n}{k} k^{k-2} p^{k-1} (1-p)^{k(n-k)+\binom{k}{2}-k+1} \right) dp$$

$$\sim \sum_{k=1}^{\log^2 n} \frac{n^k}{k!} k^{k-2} \frac{k!(k(n-k))!}{(k(n-k+1))!}$$

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$$\mathbf{E}(Z_n) \sim \int_{p=0}^{6\log n/n} \mathbf{E}(\#\text{ small isolated trees in } G_{n,p})dp$$
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$$\sim \sum_{k=1}^{\log^2 n} \frac{n^k}{k!} k^{k-2} \frac{k!(k(n-k)!}{(k(n-k+1)!}$$

So,

$$\mathbf{E}(Z_n) \sim \sum_{k=1}^{\log^2 n} \frac{1}{k^3} \sim \zeta(3).$$

Minimum Length Spanning Tree

This is most of the proof of the following:

Theorem (Frieze (1985))

$$Z_n \sim \zeta(3) \quad w.h.p.$$

Original proof not so "clean":
Remarkable integral formula is due to Janson (1995).

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With more work we have

Theorem (Cooper, Frieze, Ince, Janson, Spencer (2014))

$$\mathbf{E}(Z_n) = \zeta(3) + \frac{c_1}{n} + \frac{c_2 + o(1)}{n^{4/3}}.$$

Theorem (Cooper, Frieze, Ince, Janson, Spencer (2014))

$$\mathbf{E}(Z_n) = \zeta(3) + \frac{c_1}{n} + \frac{c_2 + o(1)}{n^{4/3}}.$$

$$c_1 = -1 - \zeta(3) - \frac{1}{2}\int_{x=0}^{\infty} \log\left(1-(1+x)e^{-x}\right)\,dx$$

and

$$c_2 = \int_{x=0}^{\infty}\left(x^{-3}\psi(x^{3/2})e^{-x^3/24} - x^{-3} - \sqrt{\frac{\pi}{8}}x^{-3/2} - \frac{1}{2}\right)\,dx$$

where if $\mathcal{B}_{\mathrm{ex}} = \int_{s=0}^{1} B_{\mathrm{ex}}(s)\,ds$ is the area under a normalized Brownian excursion,

$$\psi(t) = \mathbf{E}e^{t\mathcal{B}_{\mathrm{ex}}},$$

the moment generating function ψ of $\mathcal{B}_{\mathrm{ex}}$.

Minimum Length Spanning Tree

If we give random weights to an arbitrary r-regular graph G then under some mild expansion assumptions

Theorem (Beveridge, Frieze, McDiarmid (1998))

$$\mathbf{E}(Z_n) = \frac{n}{r}(\zeta(3) + \epsilon_r)$$

where $\epsilon_r \to 0$ as $r \to \infty$.

For example, if G is the complete bipartite graph $K_{n/2,n/2}$ then $\mathbf{E}(Z_n) \sim 2\zeta(3)$.

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Shortest Paths

Every edge e of the complete graph K_n is given a random length X_e.

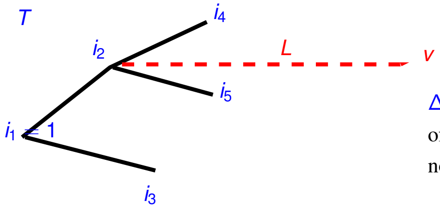
The edge lengths are independently exponentially distributed with mean 1 viz. $E(1)$ i.e. $\mathbf{Pr}(X_e \geq \lambda) = e^{-\lambda}$.

The length of a path is the sum of the lengths of its edges.

The question to be discussed is what is the length of a shortest path between two given vertices.

Let D_i be the length of a shortest path from vertex 1 to vertex i.

We can build a tree of shortest paths, adding the next closest vertex to 1 in each step.

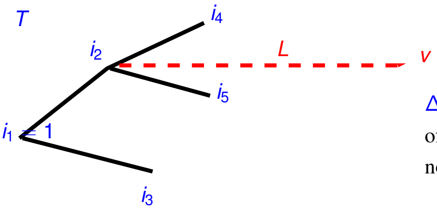


Paths in tree T are shortest paths.

Δ_v is the minimum length of a path to $v \notin T$ using one edge not in T.

If $\Delta_w = \min \Delta_v,\ v \notin T$ then $D_w = \Delta_w$.

In the above diagram we have added the 4 closest vertices to create a tree T. To find the 5th closest we compute Δ_v for each $v \notin T$ and then add the vertex that minimises Δ.



Δ_v is the minimum length of a path to $v \notin T$ using one edge not in T.

If L is the length of (i_2, v) then L is exponential conditioned on $D_{i_2} + L \geq D_{i_5}$.

So $D_{i_2} + L = D_{i_5} + E(1)$.
(Memoryless property of exponential).

So, if we add vertices to T in the order $i_1 = 1, i_2, \ldots, i_n$ then $D_{i_{k+1}} - D_{i_k}$ is the minimum of $k(n-k)$ independent $E(1)$'s.

So, if Z_i is the distance from 1 to the ith closest vertex,

$$Z_1 = 0 \text{ and } \mathbf{E}(Z_{k+1}) = \mathbf{E}(Z_k) + \frac{1}{k(n-k)}.$$

It follows that

$$\mathbf{E}(Z_n) = \frac{2}{n}\sum_{i=1}^{n-1}\frac{1}{i}.$$

Furthermore, 2 is equally likely to be the ith closest, for $i = 2, 3, \ldots, n$ and we have

$$\mathbf{E}(D_2) = \frac{1}{n-1}\sum_{i=1}^{n-1}\frac{1}{i}.$$

Theorem (Janson (1999))

Let $D_{i,j}$ be the shortest distance between i, j in the above model. Then

$$D_{1,2} \sim \frac{\log n}{n}.$$

$$\max_j D_{1,j} \sim \frac{2\log n}{n}.$$

$$\max_{i,j} D_{i,j} \sim \frac{3\log n}{n}.$$

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Background: Two-dimensional Assignment problem

Let M be a real $n \times n$ matrix of costs.

Problem: Minimise

$$\sum_{i=1}^{n} \sum_{j=1}^{n} M_{i,j} x_{i,j} \qquad \text{subject to}$$

$$\sum_{i=1}^{n} x_{i,j} = 1, \qquad j \in [n]$$

$$\sum_{j=1}^{n} x_{i,j} = 1, \qquad i \in [n]$$

$$x_{i,j} \in \{0, 1\}$$

Solvable in polynomial ($O(n^3)$) time.

Suppose now that M is a matrix of i.i.d. random variables: Let

$Z_A = Z_A(n)$ denote the random minimum value.

(a) $\mathbf{E}(Z_A) \leq 3$ – Walkup (1979) ($M_{i,j}$ is uniform $[0, 1]$).

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(e) $\mathbf{E}(Z_A(n)) = \sum_{k=1}^{n} \frac{1}{k^2}$ – Linusson and Wästlund (2004) and Nair, Prabhakar and Sharmar (2005) ($M_{i,j}$ is $Exp(1)$)

3-Dimensional Assignment Problem

Three-dimensional Axial Assignment problem.
Let M be a real $n \times n \times n$ tensor of costs.
Problem: Minimise

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} M_{i,j,k}x_{i,j,k} \qquad \text{subject to}$$

$$\sum_{i=1}^{n}\sum_{j=1}^{n} x_{i,j,k} = 1, \qquad k \in [n]$$

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Each solution has a unique one in every “plane” of the cube.

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Each solution has a unique one in every “plane” of the cube.

NP-hard – Karp (1972)

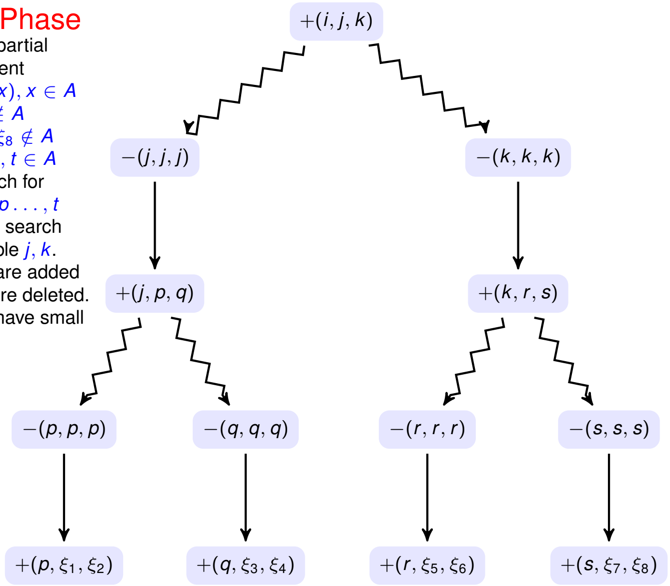
We employ a 3-phase algorithm: it has a depth parameter d.
To get a solution of value $O(n^{-(1-o(1))})$ we take $d = \epsilon \log_2 \log n$ where $\epsilon < 1/2$.

To get a feel for the algorithm, we consider $d = 2$.

Greedy Phase:
"Greedily" choose a partial assignment containing $n - n^{6/7}$ "triples" (i, j, k) at a total cost of about $n^{-6/7}$.

Main Phase

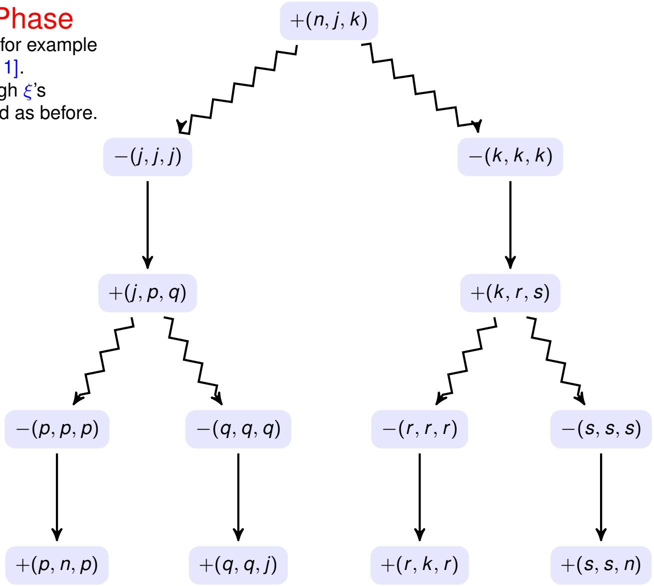
Current partial assignment is $(x, x, x), x \in A$
Here, $i \notin A$
$\xi_1, \ldots, \xi_8 \notin A$
$p, q, \ldots, t \in A$
We search for suitable $p \ldots, t$
Then we search for suitable j, k.
+triples are added
-triples are deleted.
+triples have small M-value.



3-Dimensional Assignment Problem

Final Phase

Suppose for example
$A = [n-1]$.
Not enough ξ's
to proceed as before.



3-Dimensional Assignment Problem

We have the following theorem from Frieze and Sorkin (201?):

Theorem

There is an $O(n^{3+o(1)})$ time algorithm that w.h.p. finds an assignment of value $n^{-(1-o(1))}$.

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Theorem

There is an $O(n^{3+o(1)})$ time algorithm that w.h.p. finds an assignment of value $n^{-(1-o(1))}$.

This raises the question of what is the real growth rate of the optimum value.

One simple consequence of the breakthrough paper of Johannson, Kahn, Vu (2008) is that w.h.p. there is an assignment of value $O(\log n/n)$.

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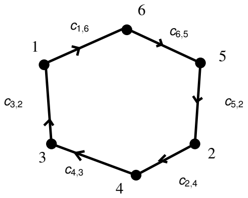
TSP with independent costs:

We are given an $n \times n$ matrix $[c_{i,j}]$ where we assume that the $c_{i,j}$ are independent uniform $[0,1]$ variables.

The aim is to compute

$$T(C) = \min \left\{ \sum_{i=1}^{n} c_{i,\pi(i)} \; : \; \pi \text{ is a } \textbf{cyclic} \text{ permutation of } [n] \right\}$$



$$\pi(1) = 6,\ \pi(2) = 4$$
$$\pi(3) = 1,\ \pi(4) = 3$$
$$\pi(5) = 2,\ \pi(6) = 5$$

TSP with independent costs:

Assignment problem The aim is to compute

$$A(C) = \min \left\{ \sum_{i=1}^{n} c_{i,\pi(i)} : \ \pi \text{ is a permutation of } [n] \right\}.$$

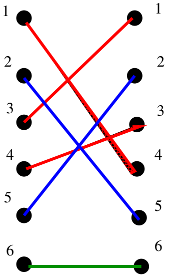
$$\frac{\pi^2}{6} \sim A(C) \leq T(C) \leq A(C) + o(1) \ w.h.p.$$

The LHS is due to Aldous (1992,2001); Nair,Prabhakar and Sharma (2006); Linusson and Wästlund (2004).
The RHS is due to Karp (1979).

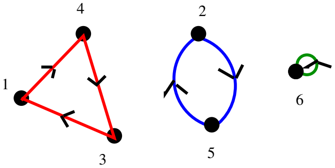
$A(C)$ is solvable in polynomial time.

TSP with independent costs:

There are two equivalent ways of viewing the assignment problem:



Minimum Weight Perfect Matching



Minimum Weight Cycle Cover

The TSP can then be thought of as finding a minimum weight cycle cover in which there is only one cycle.

TSP with independent costs:

Karp's Patching Algorithm:

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- Solve the associated assignment problem.
- Patch the cycles together to get a tour.

TSP with independent costs:

Karp's Patching Algorithm:

- Solve the associated assignment problem.
- Patch the cycles together to get a tour.

  Karp observed that if C is a matrix with i.i.d. costs then the optimal permutation is uniformly distributed and so w.h.p. the number of cycles is $\sim \log n$ – Key Observation.
- Karp showed that the cost of patching is $o(1)$ w.h.p.

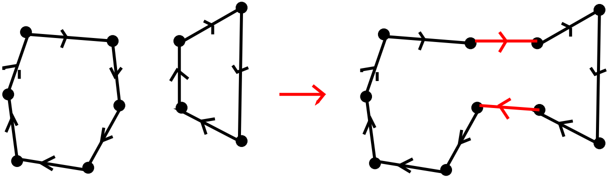

Figure: Patching two cycles

TSP with independent costs:

Theorem (Karp (1979))

W.h.p. $GAP = T(C) - A(C) = o(1)$.

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By making the cycles large before doing the patching we have

Theorem (Dyer and Frieze (1990))

W.h.p. $GAP = T(C) - A(C) = o\left(\frac{\log^4 n}{n}\right)$.

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With more care

Theorem (Frieze and Sorkin (2007))

W.h.p. $GAP = T(C) - A(C) = O\left(\frac{\log^2 n}{n}\right)$.

TSP with independent costs:

The main tool in the improvements to Karp and Steele comes from cheaply transforming the cycle cover so that each cycle has length at least $n_0 = n \log \log n / \log n$.

TSP with independent costs:

Having increased the cycle size to $n_0 = n \log \log n / \log n$ we patch the cycles together using short edges. Each patch will cost $O(\log n/n)$ and so the patching cost is $o(\log^2 n/n)$.

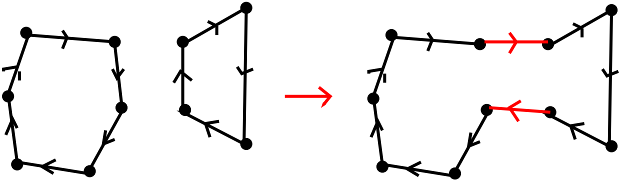

Figure: Patching two cycles

The probability we cannot patch a pair of cycles is at most

$$\left(1 - \Omega\left(\frac{\log^2 n}{n^2}\right)\right)^{\Omega(n_0^2)} = e^{-\Omega(\log^2 \log n)} = o(1/\log n).$$

TSP with independent costs:

Increasing the cycle size:

- Partition the edges into
  red edges $E_1 = \{(i,j) : c_{i,j} \leq L = K \log n\}$,
  blue edges $E_2 = \{(i,j) : c_{i,j} \in [L, 2L]\}$,
  green edges $E_3 = \{(i,j) : c_{i,j} \in [2L, 3L]\}$ and
  uncolored edges $E_4 = [n]^2 / (E_1 \cup E_2 \cup E_3)$.

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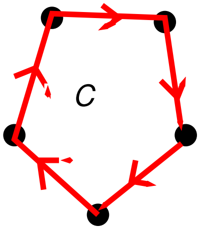
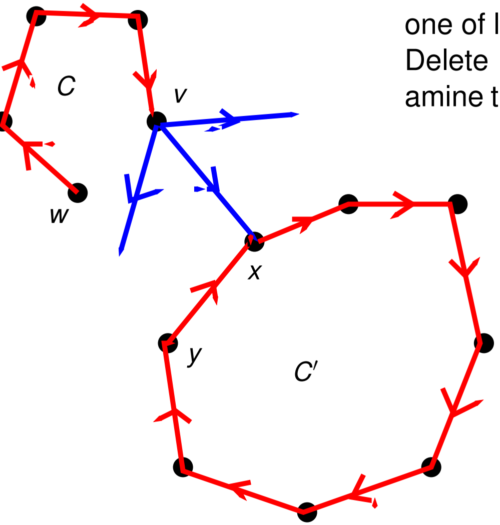
TSP with independent costs:

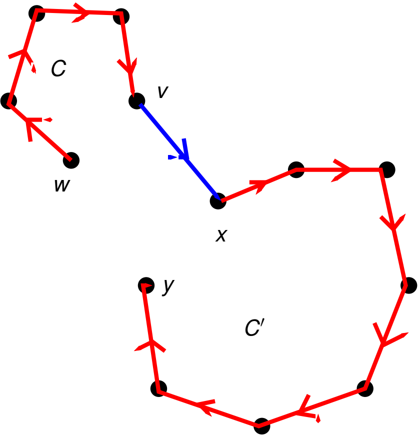


Choose some small cycle C i.e. one of length less than n_0.

TSP with independent costs:



Choose some small cycle C i.e. one of length less than n_0. Delete an edge (v, w) and examine the blue edges with tail v.



Choose some small cycle C i.e. one of length less than n_0.
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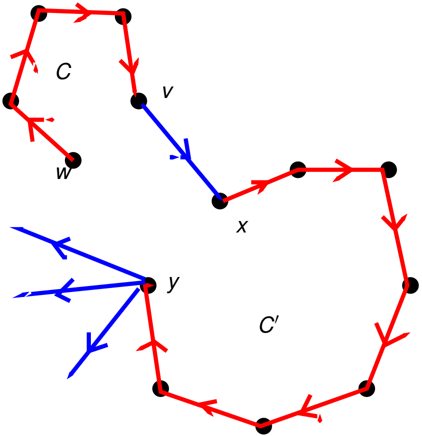
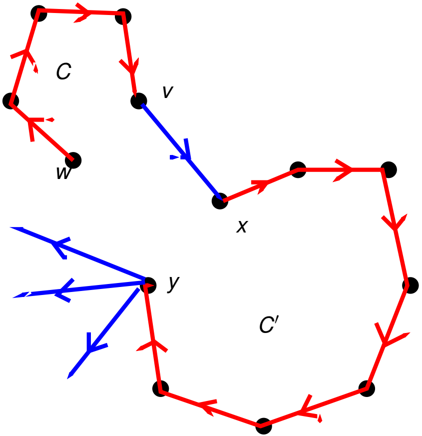
Suppose that we have blue edge (v, x) with x in a large cycle C'.

TSP with independent costs:



Choose some small cycle C i.e. one of length less than n_0.
Delete an edge (v, w) and examine the blue edges with tail v.
Suppose that we have blue edge (v, x) with x in a large cycle C'.
Delete the edge (y, x) of C'.

TSP with independent costs:

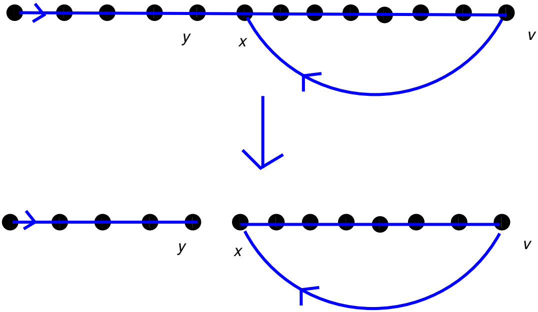


Choose some small cycle C i.e. one of length less than n_0.
Delete an edge (v, w) and examine the blue edges with tail v.
Suppose that we have blue edge (v, x) with x in a large cycle C'.
Delete the edge (y, x) of C'.

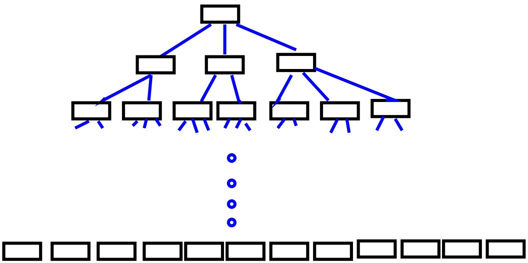
By repeating this operation, which we call a splice, we create a large number of partitions of $[n]$ into a path with initial vertex w and a collection of cycles. We call such a collection a Near

TSP with independent costs:

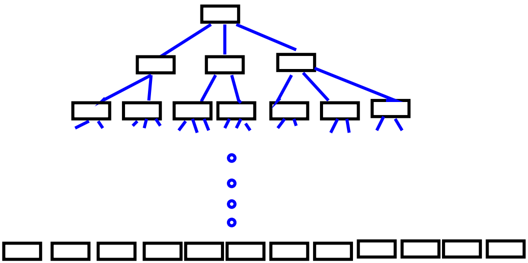
Another possibility for a splice:



In the second version of the splice we insist that the resultant path and cycle, both have length at least n_0.



In the above diagram, each rectangle is an NPD that is obtained from its parent by a splice. A node ν is allowed d_ν children. Fot the root ρ we have $d_\rho = \Theta(\log n)$ for any other node ν we have $d_\nu = \Theta(1)$. We use the cheapest available edges to extend our path. If we build this tree to depth $\sim \log n/2$ then at the bottom of the tree there will be $n^{1/2+o(1)}$ leaves.



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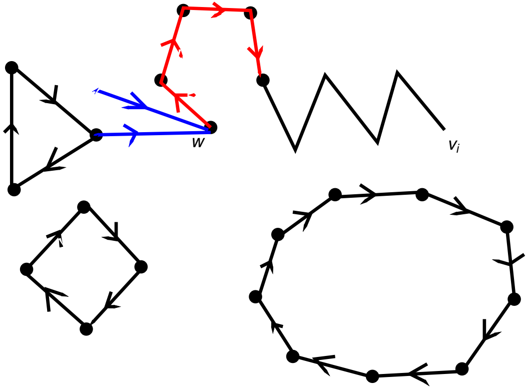
We can assume that the ith leaf corresponds to an NPD where

Now for each v_i we build a tree of NPD's where we begin by examining the edges into w.



TSP with independent costs:

Let the leaves of the ith tree have paths from $w_{i,j}$ to v_i for $j = 1, 2, \ldots, n^{1/2+o(1)}$.

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- The factor $O(\log n) \times O\left(\frac{\log n}{n}\right)$ can be replaced by $O\left(\frac{\log n}{n}\right)$ with some care.

TSP with independent costs:

Theorem (Held and Karp (1962))

In the worst-case the TSP can be solved exactly in time $O(n^2 2^n)$.

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Theorem (Frieze and Sorkin (2007))

W.h.p. the TSP can be solved exactly in $2^{O(n^{1/2})}$ *time.*

TSP with independent costs:

Let

$$I_k = \frac{(\log n)^2}{n}[2^{-k}, 2^{-k+1}].$$

W.h.p. there are $\leq c_1 2^{-(k-1)} n \log n$ non-basic variables with reduced cost in $I_k, 1 \leq k \leq k_0 = \frac{1}{2} \log_2 n$ and $\leq 2c_1 \sqrt{n} \log n$ non-basic variables with reduced cost $\leq c_1 \frac{(\log n)^2}{n^{3/2}}$.

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Thus w.h.p. we need only check at most

$$2^{2c_1\sqrt{n}\log n} \prod_{k=1}^{k_0} \sum_{t=1}^{2^k} \binom{c_1 2^{-(k-1)} n \log n}{t} = e^{O(\sqrt{n}\log^{O(1)} n)}$$

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It is natural to replace the assignment problem by that of finding a minimum weight 2-factor viz. a collection of vertex disjoint cycles that cover every vertex.

We lose control over the number of cycles in the minimum weight 2-factor. In the following theorem we only had a bound of $O(n/\log n)$ for this.

Theorem (Frieze (2004))

W.h.p. $T(C) - 2FAC(C) = o(1)$.

Contents of talk

(a) Random Discrete Structures

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  (1) Evolution
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(d) Randomly edge weighted graphs

  1. Minimum Spanning Tree
  2. Shortest Paths
  3. 3-Dimensional Assignment Problem
  4. Random Instances of the TSP with independent costs

(e) Random k-SAT

(f) Open Problems

Random k-SAT

Variables $V = \{x_1, x_2, \ldots, x_n\}$
Literals $L = \{x_1, \bar{x}_1, x_2, \bar{x}_2, \ldots, x_n, \bar{x}_n\}$
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For example $\phi(x_1) = 0$, $\phi(x_2) = \phi(x_3) = 1$ satisfies $\{\bar{x}_1, \bar{x}_2, \bar{x}_3\}$, $\{x_1, x_2, \bar{x}_3\}$, $\{\bar{x}_1, \bar{x}_2, x_3\}$.

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$\{\bar{x}_1, \bar{x}_2\}, \{x_1, \bar{x}_2\}, \{\bar{x}_1, x_2\}, \{\bar{x}_1, \bar{x}_2\}$ is unsatisfiable.

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k-SAT problem: Determine whether or not there is a satisfying assignment for I.
Solvable in polynomial time for $k \leq 2$. NP-hard for $k \geq 3$.

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So I is unsatisfiable w.h.p. if $c > 2^k \log 2$.

Random k-SAT

Conjecture: $\exists c_k$ such that if $m = cn$ then

$$\lim_{n\to\infty} \mathbf{Pr}(I \text{ is satisfiable}) = \begin{cases} 1 & c < c_k \\ 0 & c > c_k \end{cases}$$

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Now if $m = cn$ and

$$\mathbf{Pr}(Z > 0) \geq \frac{\mathbf{E}(Z)^2}{\mathbf{E}(Z^2)}$$

and if the RHS here is bounded below then Friedgut's result implies that $c \leq c_k$.

Random k-SAT

With Z equal to the number of satisfying assignments, the second moment method fails.

Achlioptas and Peres (2004) replace Z by

$$Z_1 = \sum_{\phi \text{ satisfies } I} \gamma^{H(\phi,I)}$$

where $H(\phi, I) =$ # true literals - # false literals in I for ϕ.

With a careful choice of $0 < \gamma < 1$ they proved

Theorem

If

$$c < 2^k \log 2 - (k+1)\frac{\log 2}{2} - 1 - o_k(1)$$

then I is satisfiable w.h.p.

Z_1 reduces the weight of satisfying assignments with an "excess" of true literals.

Random k-SAT

Using a more complicated random variable, based on insights from Physicists, and doing more conditioning, but still using the second moment method,

Theorem (Coja-Oghlan and Panagiotou (2012))

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Most of these find a satisfying assignment w.h.p. provided there are at most $\frac{c2^k}{k}n$ clauses, for small enough c.
A notable exception is the algorithm of Coja-Oghlan (2009) which finds a satisfying assignment w.h.p. provided there are at most $\frac{(1-\epsilon)2^k \log k}{k}n$ clauses.

Walksat

Start with the "all true" assignment: $\phi(x_j) = 1, \forall j$

Repeat

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Choose a random variable from C and change its assigned value

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Because m is small, this means that w.h.p. A is small and then $C, C' \in A$ are almost disjoint.

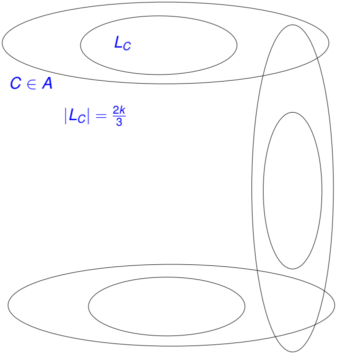
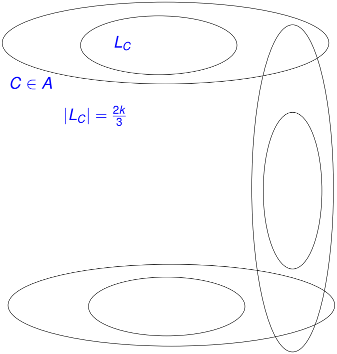


Figure: Each $C \in A$ has its own unique set of $2k/3$ literals



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Putting $\sigma_A(x) = 0$ for $\bar{x} \in \bar{V} \cap \bigcup_{C \in A} L_C$ and $\sigma_A(x) = 1$ otherwise, yields a satisfying assignment.

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Similar idea to that of Papadimitriou (1994) for 2-SAT.

Finding c_k for $k = O(1)$ is a major open problem. If we allow k to grow then things become simple: Coja-Oghlan and Frieze (2008) proved

Theorem

Suppose that $k - \log_2 n \to \infty$ and that $m = 2^k(n \ln 2 + c)$ for an absolute constant c. Then,

$$\lim_{n\to\infty} \mathbf{Pr}(I_m \textit{ is satisfiable}) = 1 - e^{-e^{-c}}$$

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Find a polynomial time algorithm that w.h.p. finds a clique of size at least $1.001 \log_2 n$ in $G_{n,1/2}$.

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Choose a p-subset $S \subseteq [n]$ and add all edges contained in S to $G_{n,1/2}$. Now ask someone else to find S.

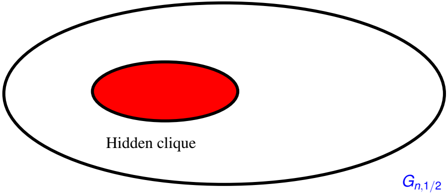

Figure: Planted Clique

Open Questions/Problems

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Choose a p-subset $S \subseteq [n]$ and add all edges contained in S to $G_{n,1/2}$. Now ask someone else to find S.

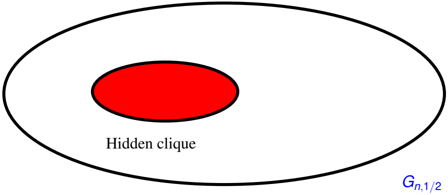

Figure: Planted Clique

If $p \gg n^{1/2}$ then enough to check vertices of high degree, Kucera (1995).

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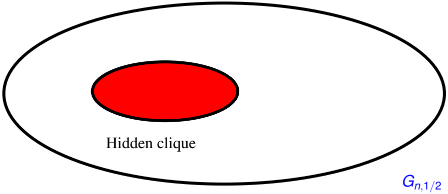

Figure: Planted Clique

If $p = O(n^{1/2})$ then spectral methods work, Alon, Krivelevich and Sudakov (1998).

Open Questions/Problems

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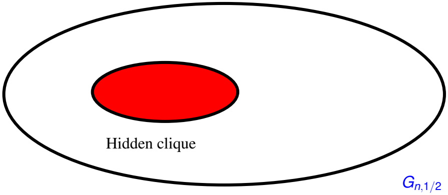
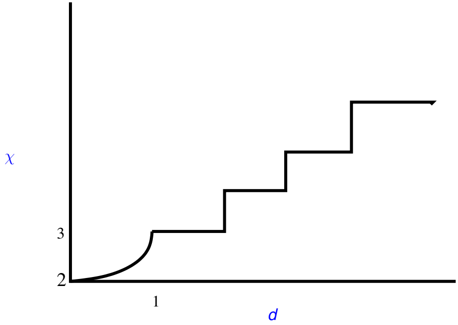


Figure: Planted Clique

If $p = o(n^{1/2})$ then there are negative results on statistical algorithms, Feldman, Grigorescu, Reyzin, Vempala and Xiao (2013).

Open Questions/Problems

Find the precise threshold for the k-colorability of the random graph $G_{n,p}$.



Open Questions/Problems

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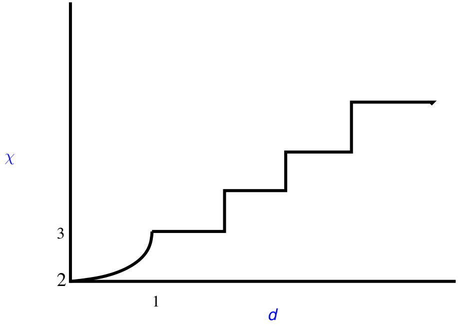

Find a polynomial time algorithm that optimally colors $G_{n,p}$ w.h.p. or prove that this is impossible under some accepted complexity conjecture.

Open Questions/Problems

Find the precise threshold for the satisfiability of random k-SAT.

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Find the precise threshold for the satisfiability of random k-SAT.

Find a polynomial time algorithm that determines the satisfiability of random k-SAT w.h.p. or prove that this is impossible under some accepted complexity conjecture.

Open Questions/Problems

Prove that $\lim_{n\to\infty} \mathbf{Pr}(G_{n,cn;3} \text{ is Hamiltonian}) = 1$ for $c > 3/2$.

Open Questions/Problems

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Construct a linear time algorithm for finding a Hamilton cycle in this model.

Open Questions/Problems

Determine whether or not solving random asymmetric TSPs with independent costs by branch and bound runs in polynomial time w.h.p. when the bound used is the assignment problem value.

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In practise, branch and bound works well on these instances.

Open Questions/Problems

Analyse the *ordinary* simplex algorithm on random instances.

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Significant results are limited to more sophisticated versions such as the shadow simplex algorithm, Borgwardt (1980).

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Led to the notion of smoothed analysis: Spielman and Teng (2004).

Open Questions/Problems

Let M be randomly chosen from the set of $n \times n$ symmetric $\{0,1\}$ matrices with $r \geq 3$ ones in each row and column. Prove that M is non-singular w.h.p.

$$\begin{vmatrix}
0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0
\end{vmatrix}$$

Open Questions/Problems

Find a heuristic for the TSP in the unit square that w.h.p. comes with n^α of the optimum, where $0 < \alpha < 1/2$ is constant.

Open Questions/Problems

Determine the constant β in the Beardwood, Halton and Hammersley theorem.

Open Questions/Problems

Determine the asymptotics for the value of a random multi-dimensional assignment problem and find asymptotically optimal heuristics.

Open Questions/Problems

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Give a uniform $[0,1]$ weight X_e to each edge of the complete 3-uniform hypergraph $H_{n:3}$. Let Z_n denote the minimum weight of a perfect matching.

Open Questions/Problems

Determine the asymptotics for the value of a random multi-dimensional assignment problem and find asymptotically optimal heuristics.

Give a uniform $[0, 1]$ weight X_e to each edge of the complete 3-uniform hypergraph $H_{n:3}$. Let Z_n denote the minimum weight of a perfect matching.

It is known that w.h.p.

$$\frac{c_1}{n} \leq Z_n \leq \frac{c_2 \log n}{n}.$$

The LHS is easy. The RHS depends on a deep result of Johansson, Kahn and Vu (2008).

Open Questions/Problems

Determine the threshold for a random subgraph of the *n*-cube to be Hamiltonian.

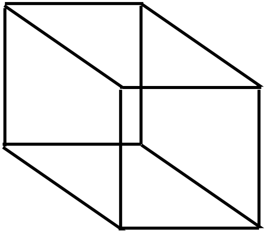

Figure: 3-cube

THANK YOU

정말 감사합니다