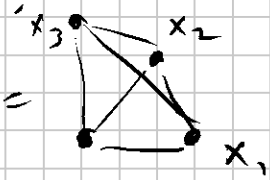
21-251 Lec 31 Notes

Note Title 11/14/2003

Tetrahedron:

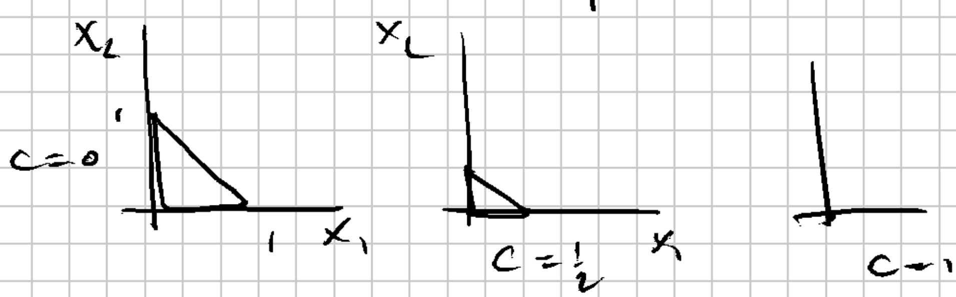
$T =$ 

Corners
"extreme points" $= \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

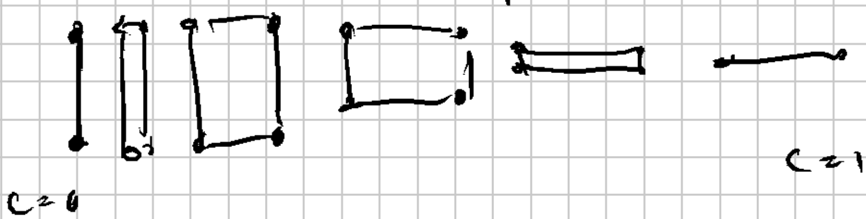
$$T = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : \begin{array}{c} x+y+z \le 1 \\ x \ge 0 \\ y \ge 0 \\ z \ge 0 \end{array} \right\}$$

$$= \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : \begin{bmatrix} 1 & 1 & 1 \\ -1 & & \\ & -1 & \\ & & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \le \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}$$

Slice T with plane $x_3 = c$, get

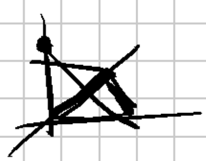


Slice T with plane $x_1 + x_2 = c$, get



This tetrahedron is 3D version of triangle:

Start with a right triangle,



add a point at $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ and fill in

(called taking the convex hull)

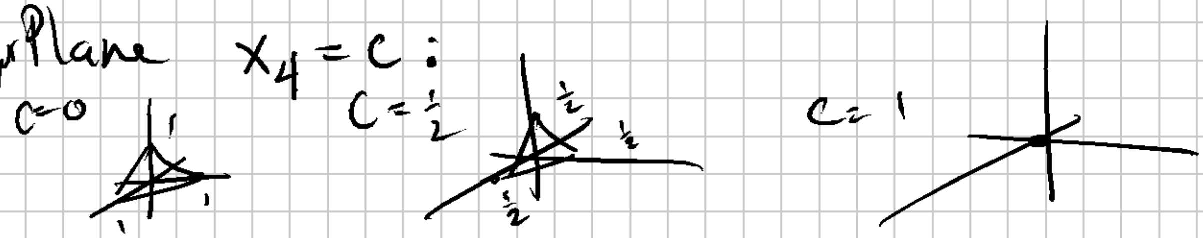
To get the 4D version of the tetrahedron, we start with tet, add a point at $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$, and fill it in.

This gives (you can show)

$$\left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} : \begin{array}{c} x_1 + x_2 + x_3 + x_4 \leq 1 \\ x_1 \geq 0 \\ x_2 \geq 0 \\ x_3 \geq 0 \quad x_4 \geq 0 \end{array} \right\}$$

I can't draw it. But I can draw slices:

hyperplane $x_4 = c$:



How about the hyperplane $x_1 + x_2 = c$? for $c=0$, so $x_1 = x_2 = 0$, get

It should have something crazy for $c = \frac{1}{2}$

extreme pts $\begin{pmatrix} 0 & 0 & 0 & .5 & .5 & .5 \\ .5 & .5 & .5 & 0 & 0 & 0 \\ 0 & .5 & 0 & 0 & .5 & 0 \\ .5 & 0 & 0 & .5 & 0 & 0 \end{pmatrix}$

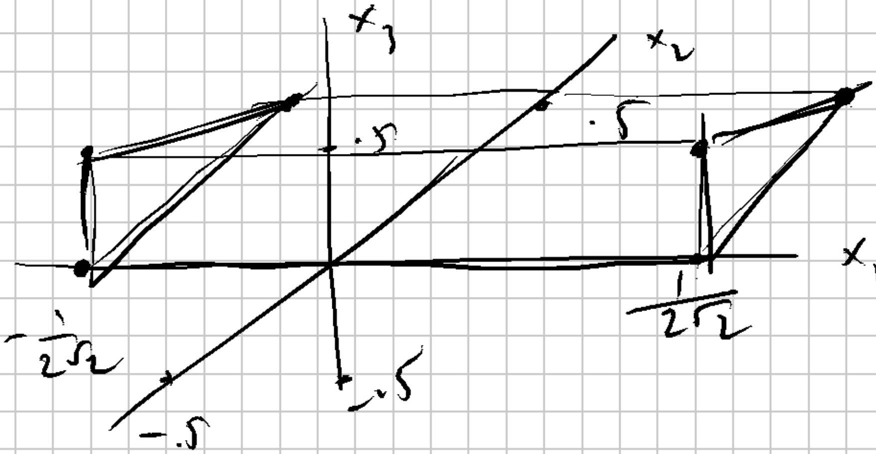
Lies in a hyperplane (3-dimensions), but not in a subspace..

① Find an orthonormal basis for hyperplane w/ $c=0$

$$U = \left\{ \begin{bmatrix} +1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

② Project onto U:

$$U^T C = \begin{bmatrix} -\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & +\frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} \\ 0 & .5 & 0 & 0 & .5 & 0 \\ .5 & 0 & 0 & .5 & 0 & 0 \end{bmatrix}$$



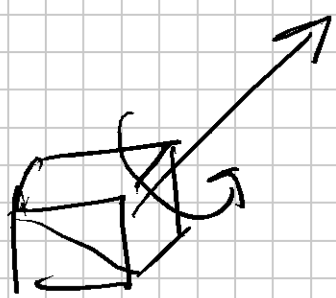
"Triangular prism"

(You will do this for a slice of the hypercube on HW.)

Last technical point on 4D Geom:

<u>Rotation</u>:

In 3Dimensions, there is always an axis of rotation:



Now we can think of rotation as a matrix transformation, where the matrix is orthogonal.

Then the axis of rotation is the <u>eigenspace</u> of the real eigenvector of that eigenspace.

In 4 Dimensions, a rotation is still an orthonormal matrix transform, but there doesn't need to be an axis of rotation.

<u>Why?</u> Look at characteristic polynomial

$$\underline{\underline{\det(A-\lambda I)}}$$

In 3D, charpoly is a cubic polynomial with real coefficients. So the imaginary roots appear in pairs, so there is always 1 real root. So there is always an eigenspace to serve as axis of rotation.

In 4D, char poly is a quartic (4th degree) polynomial. So there may be 2 pairs of complex roots, and no real eigenspace to serve as axis of rotation.