**RAMSEY THEORY FOR BALANCED GRAPHS**

**Question 1.** Let $H$ denote the edge-coloring of $K_4$ (the complete graph on 4 vertices) obtained by coloring a triangle blue and the rest of the edges red, and let $\overline{H}$ denote this same structure, with the colors switched. What is the smallest positive integer $n$ such that every edge-coloring of $K_n$ with same number of red and blue edges contains $H$ or $\overline{H}$ as a subgraph?

Let $R(k)$ denote the least positive integer $n$ such that any partition into two parts (equivalently, a red-blue coloring) of the edge set of a complete graph on $n$ vertices yields a monochromatic (edges belong to only one of the parts/colors) copy of $K_k$ as a subgraph. It is well-known (and easy to show) that $R(k) < 2^{2k}$. In the other direction, a classical probabilistic argument by Erdős shows that $R(k) \geq 2^{k/2}$. This direction is also not difficult, but neither of the constants ($2$ and $\frac{1}{2}$) that appear in the exponents have been improved for decades.

If we wish to guarantee existence of subgraphs which are not simply monochromatic, we need to assume that both color classes are well-represented. We say that a coloring of $K_n$ is $\varepsilon$-balanced if both color classes in the coloring have size at least $\varepsilon \cdot \frac{n}{2}$. A characterization of all subgraphs which appear in sufficiently large $\varepsilon$-balanced graphs was conjectured by Bollobás, and proved by Cutler and Montagh [CM08]. Asymptotically sharp bounds (ignoring constant factors in the exponent) on this problem were obtained by Fox and Sudakov [FS08], who also generalized the result to tournaments. Recent work also generalized this result to colorings with arbitrarily many color classes, as well as infinite graphs [BLM18].

In [BLM18], the problem of estimating $R_\varepsilon(H)$ (the smallest positive integer $n$ such that all $\varepsilon$-balanced red-blue colorings of $K_n$ contain $H$ or $\overline{H}$ as a subgraph) for small $H$ was also raised. In the ordinary Ramsey setting, it is known that $R(3) = 6$ and $R(4) = 18$, but after that we only have progressively weaker estimates. The smallest nontrivial case for complete graphs in the balanced setting is the one mentioned in Question 1, namely, what is $R_{1/2}(H)$ for $H$ as described there. In [BLM18], the bounds $10 \leq R_{1/2}(H) \leq 25$ were obtained. Can we improve this estimate, or even better, close the gap entirely?

After complete graphs, it is natural to turn our attention to sparser graphs:

**Question 2.** Let $D_{2k}$ denote the edge-coloring of $C_{2k}$ (the cycle on $2k$ vertices) that looks like this: red—red—blue—blue—red—red—blue—blue—\ldots—red—red—blue—blue. Estimate $R_\varepsilon(D_{2k})$. More generally, let $D$ be a cycle with an associated edge-coloring, such that $R_\varepsilon(D) < \infty$ (using [FS08], one can characterize all such $D$), then, estimate $R_\varepsilon(D)$.

Observe that, interestingly, $D_{2k}$ and $\overline{D_{2k}}$ are the same graph.

Ramsey numbers of sparse graphs, especially for cycles, is a well-studied problem [BE73]. It turns out that for such graphs, the Ramsey numbers are linear (as opposed to exponential for complete graphs) with respect to the number of vertices of the graph. In the balanced setting, ongoing work here at CMU shows that if $R_\varepsilon(H) < \infty$, and $H$ has maximum degree bounded by an absolute constant $\Delta$, then $R_\varepsilon(H) < C_{\Delta, \varepsilon} \cdot |H|$ where $C_{\Delta, \varepsilon}$ is some constant depending on $\Delta$ and $\varepsilon$. One of the simplest examples of such $H$ are cycles, and Question 2 asks for estimating the right constant determining its balanced Ramsey number. A similar question can be asked for paths as well.

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References


