In this note we define the set $\omega$ and prove its existence, assuming the axiom system $\text{ZF}^-$ (i.e., all of $\text{ZFC}$ except for $\text{AF}$ and $\text{AC}$).

A set $X$ is **inductive** if $\emptyset \in X$ and for all $x \in X$, we also have $x \cup \{x\} \in X$. The Axiom of Infinity asserts that there is an inductive set.

**Exercise.** Let $\mathcal{F}$ be a nonempty set all of whose elements are inductive sets. Show that $\bigcap \mathcal{F}$ is also an inductive set.

**Theorem ($\text{ZF}^-$).** There is a unique inductive set $\omega$ such that for every inductive set $X$, $\omega \subseteq X$.

**Proof.** It is clear that such a set $\omega$, if it exists, must be unique. Now, let $\omega$ be the following class:

$$\omega := \{ n : \forall X (X \text{ is inductive} \implies n \in X) \}.$$ 

By $\text{Inf}$, there exists some inductive set $X$, and since $\omega \subseteq X$ by definition, we see that $\omega$ is a set by $\text{Comp}$. The very definition of $\omega$ insures that $\omega$ is contained in every inductive set. It remains to verify that $\omega$ is itself inductive, which is left as an exercise. 

We emphasize that $\omega$ is **defined** as the smallest inductive set. Hence, any property of natural numbers that you might like to prove must, eventually, be derived from this definition.

**Example.** How do we prove that for all $n \in \omega$, either $\emptyset = n$ or $\emptyset \in n$? In other words, how to show that every natural number $n$ satisfies $n \geq 0$? Let

$$P := \{ n \in \omega : \emptyset = n \lor \emptyset \in n \}.$$

By $\text{Comp}$, $P$ is a set, and, by definition, $P \subseteq \omega$. We wish to show that $P = \omega$. For this, it necessary and sufficient to prove that $P$ is inductive.

$\emptyset \in P$: This is true by the definition of $P$ (and since $\emptyset \in \omega$).

If $n \in P$, then $n \cup \{n\} \in P$: Since $\omega$ is inductive, we have $n \cup \{n\} \in \omega$. Since $n \in P$, we either have $n = \emptyset$ or $\emptyset \in n$. In the first case, $n \cup \{n\} = \{\emptyset\}$ and $\emptyset \in \{\emptyset\}$, so $n \cup \{n\} = \{\emptyset\} \in P$. In the second case, $\emptyset \in n \subseteq n \cup \{n\}$, so $n \cup \{n\} \in P$ again.