LINDENBAUM’S THEOREM

The axiom system is ZFC.

Theorem (Lindenbaum). Let \( f: \mathbb{R} \to \mathbb{R} \) be an arbitrary function. Then there exist injective functions \( g, h: \mathbb{R} \to \mathbb{R} \) such that \( f = g + h \) (i.e., \( f(r) = g(r) + h(r) \) for every \( r \in \mathbb{R} \)).

Proof. Use AC to fix a bijection \( 2^{\aleph_0} \to \mathbb{R}: \alpha \mapsto r_\alpha \) (so \( \mathbb{R} = \{ r_\alpha : \alpha < 2^{\aleph_0} \} \)). A crucial observation is that for each \( \alpha < 2^{\aleph_0} \), the set \( \{ r_\beta : \beta < \alpha \} \) has cardinality \( |\alpha| \leq \alpha < 2^{\aleph_0} = |\mathbb{R}| \); i.e., it is “small” compared to the entire set \( \mathbb{R} \).

We define the values \( g(\alpha) \) and \( h(\alpha) \) for \( \alpha < 2^{\aleph_0} \) recursively. Suppose that \( \alpha \) is an ordinal \( < 2^{\aleph_0} \) and that the values \( g(\beta) \) and \( h(\beta) \) for all \( \beta < \alpha \) are already determined. Let

\[
A_\alpha := \{ g(\beta) : \beta < \alpha \} \quad \text{and} \quad B_\alpha := \{ f(\alpha) - h(\beta) : \beta < \alpha \}.
\]

To ensure that the function \( g \) is injective, we must assign to \( g(\alpha) \) a value not in \( A_\alpha \); similarly, to ensure that \( h \) is injective, we must assign to \( g(\alpha) \) a value not in \( B_\alpha \) (since \( f(\alpha) \) should be equal to \( g(\alpha) + h(\alpha) \)). Now we use our crucial observation to see that

\[
|A_\alpha \cup B_\alpha| \leq |A_\alpha| + |B_\alpha| \leq |\alpha| + |\alpha| < 2^{\aleph_0},
\]

and hence \( \mathbb{R} \neq A_\alpha \cup B_\alpha \) and \( \mathbb{R} \setminus (A_\alpha \cup B_\alpha) \neq \emptyset \). Therefore, we can fix an arbitrary choice function \( c: \mathcal{P}(\mathbb{R}) \setminus \{ \emptyset \} \to \mathbb{R} \) and set

\[
g(\alpha) := c(\mathbb{R} \setminus (A_\alpha \cup B_\alpha));
\]

\[
h(\alpha) := f(\alpha) - g(\alpha).
\]