Please use \LaTeX to type up your solutions!

The axiom system in ZF, unless explicitly indicated otherwise.

Recall that a formula is $\Sigma_1$ if it is obtained from a $\Delta_0$-formula by adding finitely many (unbounded) existential and bounded universal quantifiers. Similarly, a formula is $\Pi_1$ if it is obtained from a $\Delta_0$-formula by adding (unbounded) universal and bounded existential quantifiers. In other words, $\Pi_1$-formulas are equivalent to negations of $\Sigma_1$-formulas.

**Problem 1.** Assuming ZF, show that the statement

"$R$ is a well-ordering on a set $S$"

is equivalent both to a $\Sigma_1$-formula and to a $\Pi_1$-formula.

In the remainder of this assignment we introduce and study another inner model, called HOD.

Fix a finite tuple $\vec{p} = (p_1, \ldots, p_n)$. A set $a$ is ordinal-definable over $\vec{p}$ if there exist:

- a natural number $k \in \omega$;
- a sequence of ordinals $\alpha_1, \ldots, \alpha_k, \beta$ with $\beta > \alpha_1, \ldots, \alpha_k$ and $p_1, \ldots, p_n \in V_\beta$; and
- a $U$-formula $f \in \mathcal{F}^{k+n+1}$ with $k + n + 1$ free variables and without parameters, such that

$\{a\} = \{x \in V_\beta : V_\beta \models f(x, \alpha_1, \ldots, \alpha_k, p_1, \ldots, p_n)\}$.

Informally, a set $a$ is ordinal-definable over $\vec{p}$ if $a$ is the unique element of some $V_\beta$ that makes some $U$-formula $f$ true in $V_\beta$, where $f$ is allowed to use $p_1, \ldots, p_n$ as well as arbitrary ordinals as parameters. The class of all sets that are ordinal-definable over $\vec{p}$ is denoted by $OD(\vec{p})$. (Before moving on, you should convince yourself that $OD(\vec{p})$ is indeed a class!) We also write $\text{OD} := \text{OD}(\varnothing)$ and call sets $a \in \text{OD}$ ordinal-definable. For example, every ordinal $\alpha$ is ordinal-definable, since

$\{\alpha\} = \{x \in V_{\alpha+1} : V_{\alpha+1} \models (x = \alpha)\}$.

Similarly, for $\alpha, \beta \in \text{Ord}$, the set $\{\alpha, \beta\}$ is ordinal-definable via

$\{\{\alpha, \beta\}\} = \{x \in V_\gamma : V_\gamma \models (\alpha \in x \land \beta \in x \land \forall y \in x (y = \alpha \lor y = \beta))\}$,

where $\gamma$ is any ordinal such that $\{\alpha, \beta\} \in V_\gamma$.

**Problem 2.** Let $\varphi(x, \vec{\alpha}, \vec{p})$ be a formula with a single free variable $x$ and with parameters

$\vec{\alpha} = (\alpha_1, \ldots, \alpha_k)$ and $\vec{p} = (p_1, \ldots, p_n)$,

where $\alpha_1, \ldots, \alpha_k \in \text{Ord}$. Suppose that $a$ is the unique set satisfying $\varphi(a, \vec{\alpha}, \vec{p})$; i.e.,

$\{a\} = \{x : \varphi(x, \alpha_1, \ldots, \alpha_k, p_1, \ldots, p_n)\}$.

Show that $a \in \text{OD}(\vec{p})$.

*Hint:* Reflection Principle.

**Problem 3.** Fix a finite tuple $\vec{p} = (p_1, \ldots, p_n)$.

(a) Let $\leq^\omega \text{Ord}$ denote the class of all finite sequences of ordinals. Put a well-ordering on $\leq^\omega \text{Ord}$ and conclude that there exists a bijective class function $\Phi : \text{Ord} \to \leq^\omega \text{Ord}$ defined by a formula without parameters.
(b) Show that there is a single formula $\xi(x, y, \vec{p})$ with two free variables $x, y$ and with parameters $\vec{p}$ such that for every $\gamma \in \text{Ord}$, there is at most one set $a$ for which $\xi(a, \gamma, \vec{p})$ holds, and

$$\text{OD}(\vec{p}) = \{a : \exists \gamma \in \text{Ord} \xi(a, \gamma, \vec{p})\}.$$ 

Hint: You may use (without proof) the fact that the set $\mathcal{F}$ of all $\mathcal{U}$-formulas is countable, and that a bijection $\omega \rightarrow \mathcal{F}$ can defined by a formula without parameters.

Problem 4. Recall the Principle of Global Choice (GC):

There is a class function $gc: \mathcal{U} \rightarrow \mathcal{U}$ such that for every nonempty set $A$, $gc(A) \in A$.

Show that GC is equivalent to the assertion that there exists a finite tuple $\vec{p}$ such that every set is ordinal-definable over $\vec{p}$.

Remark. This means that while the original statement of GC involves quantification over all class functions, it is actually equivalent to a single axiom:

$$\exists \vec{p} \left( \vec{p} \text{ is a finite tuple and } \forall a (a \in \text{OD}(\vec{p})) \right).$$

Hints: To show that GC holds when $\mathcal{U} = \text{OD}(\vec{p})$, use Part (b) of Problem 3. For the other direction, take as $\vec{p}$ the tuple of parameters needed to define the class function $gc$.

A set $a$ is hereditarily ordinal-definable if $a$ and all elements of the transitive closure of $a$ are ordinal-definable. The class of all hereditarily ordinal-definable sets is denoted by $\text{HOD}$.

Problem 5. In this exercise, we establish most of the following result:

Theorem. If $\mathcal{U} \models \text{ZF}$, then $\text{HOD} \models \text{ZFC}$.

(a) Show that $\text{HOD}$ is a transitive class and $\text{Ord} \subseteq \text{HOD}$.

(b) Pick and prove any two of the following claims:

- $\text{HOD}$ satisfies the Union Axiom;
- $\text{HOD}$ satisfies the Powerset Axiom;
- $\text{HOD}$ satisfies the Comprehension Schema;
- $\text{HOD}$ satisfies the Replacement Schema.

Hint: Problem 2 is your friend.

(c) Show that $\text{HOD}$ satisfies AC.

Caution: You have to be careful because $\text{HOD}$ might fail to satisfy GC.