The axiom system in \( ZF^- \). For some of the problems, extra axioms are indicated in parentheses. Please use \( \LaTeX \) to type up your solutions!

**Problem 1 (AC).** In this problem, \( \mathbb{R}^3 := \mathbb{R} \times \mathbb{R} \times \mathbb{R} \) denotes, as usual, the 3-dimensional Euclidean space, i.e., the set of all ordered triples of real numbers.\(^1\) Show that there is a set \( L \) of pairwise disjoint lines in \( \mathbb{R}^3 \) such that \( \bigcup L = \mathbb{R}^3 \) and no two lines in \( L \) are parallel to each other.

*Hint:* This is a course in set theory, not geometry.

**Problem 2 (AC).**

(a) Let \( (A, <) \) be a linearly ordered set. Show that there is a subset \( B \subseteq A \) such that \( B \) is well-ordered by \( < \) and cofinal in \( A \), meaning that for all \( x \in A \), there is \( y \in B \) with \( x \leq y \).

(b) Let \( F \) be a set that is linearly ordered by the subset relation \( \subset \). Suppose that \( \kappa \) is a cardinal such that \( |A| < \kappa \) for all \( A \in F \). Show that \( |\bigcup F| \leq \kappa \).

(c) Give an example of a set \( F \) that is linearly ordered by the subset relation \( \subset \) such that
\[
|\bigcup F| > \sup\{|A| : A \in F\}.
\]

**Problem 3.** The purpose of this exercise is to prove the following theorem of Tarski:

**Theorem** (Tarski). *Suppose that every infinite set \( X \) satisfies \( X \times X \approx X \). Then AC holds.*

Assume that every infinite set \( X \) satisfies \( X \times X \approx X \). Let \( A \) be an arbitrary set. Our goal is to show that \( A \) can be well-ordered, thus proving AC. If \( A \) is finite (i.e., if there is a bijection \( A \to n \) for some \( n < \omega \)), then \( A \) can be well-ordered, so let \( A \) be infinite. Recall that by Hartogs’s theorem, there is a cardinal \( \kappa \) such that \( \kappa \not\leq A \).

(a) Use the assumption of Tarski’s theorem to show that \( A \times \kappa \not\leq A \cup \kappa \).

(b) Use an arbitrary injection \( f : A \times \kappa \to A \cup \kappa \) to construct an injection \( A \to \kappa \).

(c) Conclude that \( A \) can be well-ordered.

**Problem 4 (AC).** For a set \( X \), let \([X]^{\leq \omega}\) denote the set of all countable subsets of \( X \).

(a) Show that for every infinite cardinal \( \kappa \), \( \kappa^{\aleph_0} = |[\kappa]^{\leq \omega}| \).

(b) Show that for every \( n < \omega \), \( \aleph_{n+1}^{\aleph_0} = \aleph_{n+1} \otimes \aleph_{n}^{\aleph_0} \).

*Hint:* cf(\( \aleph_{n+1}^{\aleph_0} \)) > \( \omega \).

(c) Conclude that for every \( n < \omega \), \( \aleph_{n+1}^{\aleph_0} = \max\{\aleph_{n}, 2^{\aleph_0}\} \).

**Problem 5.** Prove the following “uniform” version of Cantor’s theorem:

**Theorem** (Uniform Cantor’s theorem). *There is a class function \( \Phi : \mathcal{U} \times \mathcal{U} \to \mathcal{U} \times \mathcal{U} \) such that, given any set \( X \) and a function \( f : \mathcal{P}(X) \to X \), we have \( \Phi(X, f) = (\mathcal{A}, \mathcal{B}) \), where \( \mathcal{A} \) and \( \mathcal{B} \) are two distinct subsets of \( X \) such that \( f(\mathcal{A}) = f(\mathcal{B}) \) (thus witnessing the non-injectivity of \( f \)).

*Caution:* We are not assuming AC!

*Hint:* Use \( f \) to construct a class function \( \text{Ord} \to X \). This class function cannot be injective.

---

\(^{1}\)I couldn’t bring myself to write \( ^3 \mathbb{R} \) instead.