Please use \LaTeX to type up your solutions!

Problem 1. Let $\alpha$ be an ordinal. Show that every element of $\alpha$ is also an ordinal.

Problem 2. Prove that $\omega$ is an ordinal.

Problem 3 (Increasing maps between ordinals). Let $(\mathbb{A}, <_{\mathbb{A}})$ and $(\mathbb{B}, <_{\mathbb{B}})$ be linearly ordered sets. A function $f: \mathbb{A} \to \mathbb{B}$ is \textbf{strictly increasing} if
\[ \forall x, y \in \mathbb{A} \quad x <_{\mathbb{A}} y \implies f(x) <_{\mathbb{B}} f(y). \]
A function $f: \mathbb{A} \to \mathbb{B}$ is an \textbf{(order-)isomorphism} from $(\mathbb{A}, <_{\mathbb{A}})$ to $(\mathbb{B}, <_{\mathbb{B}})$ if $f$ is bijective and
\[ \forall x, y \in \mathbb{A} \quad x <_{\mathbb{A}} y \iff f(x) <_{\mathbb{B}} f(y). \]
We say that $(\mathbb{A}, <_{\mathbb{A}})$ and $(\mathbb{B}, <_{\mathbb{B}})$ are \textbf{(order-)isomorphic} if there is an isomorphism between them.

(a) Let $(\mathbb{A}, <_{\mathbb{A}})$ and $(\mathbb{B}, <_{\mathbb{B}})$ be linearly ordered sets. Show that a bijection $f: \mathbb{A} \to \mathbb{B}$ is an isomorphism if and only if it is strictly increasing.

In what follows, we always treat ordinals as ordered sets equipped with the usual ordering $<$ on ordinals (i.e., $\alpha < \beta$ if and only if $\alpha \in \beta$).

(b) Let $\alpha, \beta \in \text{Ord}$ and let $f: \alpha \to \beta$ be a strictly increasing function. Show that $\gamma \in f(\gamma)$ for all $\gamma < \alpha$ and deduce that $\alpha \leq \beta$.

Hint: Suppose there is some $\gamma$ such that $f(\gamma) < \gamma$ and let $\gamma$ be the smallest such. What can you say about $f(f(\gamma))$?

(c) Show that if two ordinals $\alpha$ and $\beta$ are order-isomorphic, then $\alpha = \beta$ and the only isomorphism is the identity map.

Problem 4 (Supremums). Let $X$ be a set of ordinals.

(a) Show that the class $X^+ := \{ \alpha \in \text{Ord} : \forall \beta \in X \ (\beta \leq \alpha) \}$ is nonempty.

Hint: Show that $\text{Ord} \setminus X^+$ is a set.

Since $X^+$ is a nonempty class of ordinals, $X^+$ has a least element. The least element of $X^+$ is called the \textbf{supremum} of $X$ and is denoted by $\text{sup} \ X$. To wit, the supremum of $X$ is the least ordinal greater than or equal to every element of $X$.

(b) Show that $\bigcup X$ is an ordinal.

(c) Conclude that $\text{sup} \ X = \bigcup X$.

Problem 5. Let $\alpha$ be a nonzero ordinal. Show that $\alpha$ is a limit ordinal if and only if $\alpha = \text{sup} \ \alpha$.