GOODSTEIN’S THEOREM

In this note we shall use infinitary techniques to prove an extremely surprising result from elementary number theory. To state it, we need a few definitions. Given a positive integer $n$ and an integer $b \geq 2$, a base-$b$ expansion of $n$ is an expression of the form

$$n = b^k \cdot d_k + b^{k-1} \cdot d_{k-1} + \cdots + b \cdot d_1 + d_0,$$

where $k \in \mathbb{N}$ and $d_0, \ldots, d_k$ are integers between 0 and $b - 1$, called the digits of the expansion. It is a well-known and easy observation that every $n \geq 1$ has a base-$b$ expansion for all $b \geq 2$; furthermore, if we additionally require that $d_k \neq 0$, then the base-$b$ expansion of $n$ is unique. For instance, the base-2, base-3, base-6, and base-10 expansions of 100 are

$$100 = 2^6 + 2^5 + 2^2 = 3^4 + 3^2 \cdot 2 + 1 = 6^2 \cdot 2 + 6 \cdot 4 + 4 = 10^2.$$ 

(For better readability we have omitted the terms whose corresponding digits are 0 and simplified the expressions such as $b^j \cdot 1$ to $b^j$.) The hereditary base-$b$ expansion of $n$ is obtained from the ordinary base-$b$ expansion by replacing every exponent by its own base-$b$ expansion, then replacing all the exponents inside the exponents by their base-$b$ expansions, &c. This idea is best understood through an example. Here’s the hereditary base-2 expansion of 100:

$$100 = 2^{2^2 + 2} + 2^{2^2 + 1} + 2^2. \quad (*)$$

And here’s the hereditary base-2 expansion of 1000000:

$$1000000 = 2^{2^2 + 2^2 + 2} + 2^{2^2 + 2} + 2^{2^2 + 1} + 2^{2^2 + 2^2 + 2} + 2^{2^2 + 1 + 2^2 + 2} + 2^{2^2 + 1} + 2^{2^2 + 2^2}.$$ 

A Goodstein sequence is a sequence of natural numbers obtained as follows. Start with an arbitrary positive integer $n_2$ (you will see in a moment why we start indexing with 2 instead of 0 or 1). To obtain the next member of the sequence, $n_3$, proceed as follows: Compute the hereditary base-2 expansion of $n_2$; then replace all 2s by 3s. Call the result $m_2$ and let $n_3 := m_2 - 1$. For instance, if $n_2 = 100$, then, using $(*)$, we obtain

$$m_2 = 3^{3^{3^3 + 3} + 3^{3^3 + 1} + 3} = 228767924549637, \text{ hence } n_3 = 228767924549636.$$ 

In general, when the value $n_b$ for some $b \geq 2$ has been determined, we proceed as follows. If $n_b = 0$, then the sequence terminates. Otherwise, we compute the hereditary base-$b$ expansion of $n_b$ and replace all $b$s by $(b + 1)$s, calling the result $m_b$. Then $n_{b+1} := m_b - 1$. For example, the hereditary base-3 expansion of 228767924549636 is

$$228767924549636 = 3^{3^{3^3 + 3} + 3^{3^3 + 1} + 3^2 \cdot 2 + 3 \cdot 2 + 2},$$ 

and thus, if we start with $n_2 = 100$, then

$$m_3 = 4^{4^{4^4 + 4} + 4^4 + 4^2 \cdot 2 + 4 \cdot 2 + 2}, \text{ and so } n_4 = m_3 - 1 \approx 3.4 \cdot 10^{156}.$$ 

As you can see, Goodstein sequences typically experience explosive growth. This is what makes the following result so shocking:

**Theorem** (Goodstein). Regardless of what natural number $n_2$ a Goodstein sequence starts with, it terminates after finitely many steps.
Before we give a proof of Goodstein’s theorem, it is worthwhile to consider some small examples. Say, if \( n_2 = 3 \), then the corresponding Goodstein sequence terminates on stage 7:
\[
\begin{align*}
n_2 &= 3, \quad n_3 = 3, \quad n_4 = 3, \quad n_5 = 2, \quad n_6 = 1, \quad n_7 = 0.
\end{align*}
\]
On the other hand, already for \( n_2 = 4 \), the sequence seems to increase steadily:
\[
\begin{align*}
n_2 &= 4, \quad n_3 = 26, \quad n_4 = 41, \quad n_5 = 60, \quad n_6 = 83, \quad n_7 = 109, \ldots \end{align*}
\]
As Goodstein’s theorem predicts, this sequence does terminate eventually... namely on step \( 3 \cdot 2^{402653211} - 1 \).

And with \( n_2 = 5 \), the sequence reaches 0 after the number of steps that is much, much greater than \( 10^{10^{10000}} \).

**Proof of Goodstein’s theorem.** The idea can be summarized in just two sentences: On step \( b \), instead of merely replacing each \( b \) by \((b + 1)\), go “all in” and replace each \( b \) by the ordinal \( \omega \). This operation will produce a strictly decreasing sequence of ordinals (due to subtracting 1 on each stage), which cannot be infinite.

Now to fill in some details. For each \( b \geq 2 \), let \( F_b: \omega \to \text{Ord} \) be the function defined as follows: To determine \( F_b(n) \), first compute the hereditary base-\( b \) expansion of \( n \) and then replace each \( b \) by \( \omega \), interpreting the addition, multiplication, and exponentiation as the corresponding operations of ordinal arithmetic. For instance, using (*), we see that
\[
\begin{align*}
F_2(100) &= \omega^{\omega^2} + \omega^{\omega^2 + 1} + \omega^\omega \quad \text{and} \quad F_3(100) = \omega^{\omega + 1} + \omega^2 \cdot 2 + 1.
\end{align*}
\]

It is important to keep in mind that the order of addition and multiplication matters (since the operations of ordinal arithmetic are not commutative).

**Lemma.** For each \( b \geq 2 \), the function \( F_b: \omega \to \text{Ord} \) is strictly increasing.

*Proof.* It is enough to show that for all \( n \in \mathbb{N} \), we have \( F_b(n + 1) > F_b(n) \). The proof is by induction on \( n \), so suppose that for all \( m < n \), we have \( F_b(m + 1) > F_b(m) \). Let the base-\( b \) expansion of \( n \) be
\[
n = \sum_{i=0}^{k} b^i \cdot d_i.
\]
Set \( d_{k+1} := 0 \) and let \( j \) be the least index such that \( d_j \neq b - 1 \). The standard elementary school addition algorithm shows that the base-\( b \) expansion of \( n + 1 \) is given by
\[
n + 1 = \sum_{i=j+1}^{k+1} b^i \cdot d_i + b^j \cdot (d_j + 1).
\]
Therefore,
\[
F_b(n) = \sum_{i=j+1}^{k+1} \omega F_b(i) \cdot d_i + \omega F_b(j) \cdot d_j + \sum_{i=0}^{j-1} \omega F_b(i) \cdot (b - 1);
\]
and
\[
F_b(n + 1) = \sum_{i=j+1}^{k+1} \omega F_b(i) \cdot d_i + \omega F_b(j) \cdot (d_j + 1).
\]
(We emphasize that in the above expressions, the summation indicated by “\( \sum \)” should be taken in decreasing order.) It is now clear that to prove \( F_b(n + 1) > F_b(n) \), it suffices to argue that
\[
\omega F_b(j) > \sum_{i=0}^{j-1} \omega F_b(i) \cdot (b - 1).
\]
If \( j = 0 \), then \( \omega^{F_b(j)} = 1 > 0 \). On the other hand, if \( j > 0 \), then, since, by the choice of \( n \), the function \( F_b \) is strictly increasing on \( \{0, \ldots, n\} \), we have

\[
\sum_{i=0}^{j-1} \omega^{F_b(i)} \cdot (b - 1) = \omega^{F_b(j-1)} \cdot (b - 1) + \omega^{F_b(j-2)} \cdot (b - 1) + \cdots + \omega^{F_b(0)} \cdot (b - 1)
\]

\[
< \omega^{F_b(j-1)} \cdot \omega = \omega^{F_b(j-1)+1} \leq \omega^{F_b(j)},
\]
as desired.

Now consider an arbitrary Goodstein sequence \( n_2, n_3, \ldots \). We associate to it a sequence of ordinals \( \alpha_2, \alpha_3, \ldots \) by setting \( \alpha_b := F_b(n_b) \). By definition, the number \( n_{b+1} + 1 \) is obtained from the base-\( b \) expansion of \( n_b \) by replacing all \( b \)'s by \( (b + 1) \)'s. Hence,

\[
\alpha_b = F_b(n_b) = F_{b+1}(n_{b+1} + 1) > F_{b+1}(n_{b+1}) = \alpha_{b+1}.
\]

Hence, the sequence \( \alpha_2, \alpha_3, \ldots \) is strictly increasing, so it must be finite.

**Exercise (Cantor normal form).** Let \( \alpha \) be an ordinal. Show that \( \alpha \) can be expressed in “base-\( \omega \),” in the following sense: There exist a unique finite sequence of ordinals \( \beta_1 > \cdots > \beta_k \) and a unique sequence of nonzero natural numbers \( d_1, \ldots, d_k \) such that

\[
\alpha = \omega^\beta_1 \cdot d_1 + \cdots + \omega^\beta_k \cdot d_k.
\]

We remark that our use of infinitary techniques to prove Goodstein’s theorem is not coincidental. It was shown by Kirby and Paris that Goodstein’s theorem is a result in elementary number theory that cannot be proven using the so-called *Peano Arithmetic*, which is a standard axiomatization of the natural numbers.