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Solve exactly six of the given problems. In the second column of the table below, write “Yes” or “No” next to each problem to indicate which six problems you wish to be graded.

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The axiom system is ZF, unless explicitly stated otherwise.

No calculators, books, notes, &c. are allowed. Please justify all your answers.

Good luck!
Problem 1. (20 pts.)
Suppose that $X$ and $Y$ are sets such that $X \times Y = X$. Show that $X = \emptyset$.

Suppose, toward a contradiction, that $X \neq \emptyset$ and let $x \in X$ be an element with the least rank. Since $X = X \times Y$, $x$ must be an ordered pair of the form $x = (x', y)$ with $x' \in X$ and $y \in Y$. In other words, $x = \{x', \{x', y\}\}$. But then
\[
\text{rank}(x') < \text{rank}(\{x'\}) < \text{rank}(x),
\]
contradicting the choice of $x$. 
Problem 2. (20 pts.)
Show that if $\alpha$ is a limit ordinal, then there is $\beta \in \text{Ord}$ such that $\alpha = \omega \cdot \beta$.

Consider the set $S := \{\gamma \leq \alpha : \alpha \geq \omega \cdot \gamma\}$, and let $\beta := \sup S$. We claim that $\beta \in S$, i.e., $\beta$ is the maximum element of $S$. Indeed, we have

$$\omega \cdot \beta = \omega \cdot \sup S = \sup \{\omega \cdot \gamma : \gamma \in S\} \leq \alpha,$$

by the definition of $S$. Now, since $\beta$ is the maximum element of $S$, we have $\beta + 1 \notin S$, i.e., $\alpha < \omega \cdot (\beta + 1) = \omega \cdot \beta + \omega$. To summarize, we have

$$\omega \cdot \beta \leq \alpha < \omega \cdot \beta + \omega.$$

This means that $\alpha = \omega \cdot \beta + n$ for some $n \in \omega$. Since $\alpha$ is a limit, we conclude that $n = 0$, as desired.
Problem 3. (7+13 pts.)
Let $\Phi : \text{Ord} \to \mathcal{U}$ and $\Psi : \text{Ord} \to \mathcal{U}$ be class functions such that $\text{ran}(\Phi) = \text{ran}(\Psi)$.

(a) Show that for every $\alpha \in \text{Ord}$, there is some $\beta \in \text{Ord}$ such that $\Phi[\alpha] \subseteq \Psi[\beta]$ and $\Psi[\alpha] \subseteq \Phi[\beta]$.

(b) Show that there exists a nonzero ordinal $\alpha$ such that $\Phi[\alpha] = \Psi[\alpha]$.

(a) Let $\mathcal{C} := \text{ran}(\Phi) = \text{ran}(\Psi)$ and define, for each $x \in \mathcal{C}$,

$$\gamma_x := \min\{\gamma \in \text{Ord} : \Phi(\gamma) = x\} \quad \text{and} \quad \delta_x := \min\{\delta \in \text{Ord} : \Psi(\delta) = x\}.$$ 

Now let $\alpha \in \text{Ord}$ and define

$$\beta_0 := \sup\{\delta_x : x \in \Phi[\alpha]\} + 1 \quad \text{and} \quad \beta_1 := \sup\{\gamma_x : x \in \Psi[\alpha]\} + 1.$$ 

By definition, $\Phi[\alpha] \subseteq \Psi[\beta_0]$ and $\Psi[\alpha] \subseteq \Phi[\beta_1]$. Setting $\beta := \max\{\beta_0, \beta_1\}$ finishes the proof.

(b) Using part (a), for every $\alpha \in \text{Ord}$, let

$$\beta(\alpha) := \min\{\beta \in \text{Ord} : \Phi[\alpha] \subseteq \Psi[\beta] \text{ and } \Psi[\alpha] \subseteq \Phi[\beta]\}.$$ 

Recursively define a sequence of ordinals $\alpha_n$, $n \in \omega$, as follows:

$$\alpha_0 := 1, \quad \alpha_{n+1} := \beta(\alpha_n) \text{ for all } n \in \omega.$$ 

We claim that $\alpha := \sup\{\alpha_n : n \in \omega\}$ is as desired. Indeed, since $\alpha = \bigcup\{\alpha_n : n \in \omega\}$, we have

$$\Phi[\alpha] = \bigcup\{\Phi[\alpha_n] : n \in \omega\} \subseteq \bigcup\{\Psi[\alpha_{n+1}] : n \in \omega\} \subseteq \bigcup\{\Psi[\alpha_n] : n \in \omega\} = \Psi[\alpha],$$

and, similarly,

$$\Psi[\alpha] = \bigcup\{\Psi[\alpha_n] : n \in \omega\} \subseteq \bigcup\{\Phi[\alpha_{n+1}] : n \in \omega\} \subseteq \bigcup\{\Phi[\alpha_n] : n \in \omega\} = \Phi[\alpha].$$
Problem 4. (20 pts.) For this problem, assume AC.

Show that there exists a cardinal $\kappa$ with $\text{cf}(\kappa) = \aleph_1$ and $\kappa = \aleph_\kappa$.

Recursively define cardinals $\lambda_\alpha$ for $\alpha \leq \aleph_1$ as follows:

\[
\begin{align*}
\lambda_0 & := 0; \\
\lambda_{\beta+1} & := (\aleph_{\lambda_\beta})^+; \\
\lambda_\alpha & := \sup\{\lambda_\gamma : \gamma < \alpha\} \text{ if } \alpha \text{ is a limit.}
\end{align*}
\]

We claim that $\kappa := \lambda_{\aleph_1}$ is as desired. Indeed, the function

\[\aleph_1 \to \kappa : \alpha \mapsto \lambda_\alpha\]

is clearly strictly increasing and cofinal in $\kappa$, hence $\text{cf}(\kappa) = \text{cf}(\aleph_1) = \aleph_1$. (We made sure that this function is strictly increasing by setting $\lambda_{\beta+1}$ to be $(\aleph_{\lambda_\beta})^+$ and not just $\aleph_{\lambda_\beta}$. ) Since it is always the case that $\aleph_\kappa \geq \kappa$, it remains to show that $\aleph_\kappa \leq \kappa$. But we have

\[\aleph_\kappa = \sup\{\aleph_{\lambda_\alpha} : \alpha < \aleph_1\} \leq \sup\{\lambda_{\alpha+1} : \alpha < \aleph_1\} = \kappa,\]

and so the proof is complete.
Problem 5. (20 pts.) For this problem, assume AC.

Show that there exists a function \( f : \mathbb{R} \to \mathbb{R} \) such that for any two real numbers \( a, b \in \mathbb{R} \) with \( a < b \), the image of the open interval \((a; b)\) under \( f \) is equal to \( \mathbb{R} \), i.e.,

\[
f[(a; b)] = \mathbb{R}.
\]

**First solution.** Using AC, fix a bijection

\[
2^\mathbb{R} \to \{(a, b, c) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R} : a < b\} : \alpha \mapsto (a_\alpha, b_\alpha, c_\alpha).
\]

Next we recursively define an injective function \( 2^\mathbb{R} \to \mathbb{R} : \alpha \mapsto x_\alpha \) so that for all \( \alpha < 2^\mathbb{R} \),

\[
a_\alpha < x_\alpha < b_\alpha.
\]

Let \( \alpha < 2^\mathbb{R} \) and suppose that for all \( \beta < \alpha \), \( x_\beta \) has been determined. Since the set

\[
X_\alpha := \{x_\beta : \beta < \alpha\}
\]

has cardinality \(|\beta| = 2^\mathbb{R} = |(a_\alpha; b_\alpha)|\), the can use AC to make \( x_\alpha \) be an arbitrary point in \((a_\alpha; b_\alpha)\)\(\setminus X_\alpha\).

Once the function \( 2^\mathbb{R} \to \mathbb{R} : \alpha \mapsto x_\alpha \) as above has been chosen, we can simply set

\[
f(x) := \begin{cases} c_\alpha & \text{if } x = x_\alpha; \\ 0 & \text{if } x \notin \{x_\alpha : \alpha < 2^\mathbb{R}\}. \end{cases}
\]

**Second solution.** Let \( E \) be the equivalence relation on \( \mathbb{R} \) given by

\[
x E y \iff x - y \in \mathbb{Q}.
\]

For each \( x \in \mathbb{R} \), let \([x]_E\) denote the \( E \)-equivalence class of \( x \); in other words,

\[
[x]_E = \{x + q : q \in \mathbb{Q}\}.
\]

This shows that each \( E \)-equivalence class is countable, and thus \(|\mathbb{R}/E| = 2^\mathbb{R}\). Hence, we can fix a bijection \( h : \mathbb{R}/E \to \mathbb{R} \). Then the following function \( f \) is as desired:

\[
f(x) := h([x]_E).
\]

Indeed, every \( E \)-equivalence class intersects every open interval, so for all \( a < b \) and \( c \in \mathbb{R} \), we can find \( x \in (a; b) \) such that \([x]_E = h^{-1}(c)\), and thus \( f(x) = c \).

**Third solution.** This solution does not use AC, and we only sketch it, leaving the details to the reader. Call a real number \( x \) **pretty** if its base-10 expansion is of the following form:

\[
x = \pm \underbrace{N}_{\text{integer part}} . \underbrace{\ldots \ldots}_{{\text{some digits}}} \underbrace{9}_{\text{either 0 or 1}} \underbrace{d_0}_{\text{some digits, not 9}} \underbrace{9}_{\text{some digits, not 9}} \underbrace{d_1 \ldots d_k}_{\text{some digits, not 9}} \underbrace{9}_{\text{some digits, not 9}} \underbrace{d_{k+1}d_{k+2} \ldots \ldots \ldots}_{\text{}}
\]

For a pretty number \( x \) as above, define \( f(x) \) to be the number whose base-9 expansion is:

\[
f(x) := \begin{cases} + & \text{if } d_0 = 0 \\ - & \text{if } d_0 = 1 \end{cases} \underbrace{d_1 \ldots d_k}_{\text{}} \underbrace{d_{k+1}d_{k+2} \ldots \ldots \ldots}_{\text{}}
\]

If \( x \) is not pretty, then set \( f(x) \) to be anything, say 0. It is not hard to verify that for all \( a < b \) and \( c \in \mathbb{R} \), there is a pretty number \( x \in (a; b) \) such that \( f(x) = c \), as desired.
Problem 6. (5+15 pts.)

(a) State the Reflection Principle.

(b) Recall that ZF consists of infinitely many axioms. Assuming ZF is consistent, show that infinitely many axioms are necessary; i.e., there is no finite list \( \varphi_1, \ldots, \varphi_n \) of formulas (with no free variables and without parameters) such that ZF is equivalent to \( \varphi_1 \land \ldots \land \varphi_n \).

*Hint:* Assuming that \( U \models ZF \) and that ZF is equivalent to \( \varphi_1 \land \ldots \land \varphi_n \), construct an infinite decreasing sequence of ordinals in \( U \).

(a) Theorem (Reflection Principle). Let \( C = \bigcup \{ C_\alpha : \alpha \in \text{Ord} \} \) be a stratified class and let \( \varphi_1, \ldots, \varphi_n \) be a finite list of formulas without parameters. Then there is an ordinal \( \beta \) such that the formulas \( \varphi_1, \ldots, \varphi_n \) are absolute between \( C_\beta \) and \( C \).

(b) Let \( \varphi_1, \ldots, \varphi_n \) be a finite list of formulas such that ZF is equivalent to \( \varphi_1 \land \ldots \land \varphi_n \). For brevity, define \( \varphi := \varphi_1 \land \ldots \land \varphi_n \); thus, ZF is equivalent to a single formula \( \varphi \).

**Claim.** Suppose that \( U \models ZF \). Then there is an ordinal \( \beta \) in \( U \) such that \( V_\beta \models ZF \).

**Proof.** Since \( ZF \leftrightarrow \varphi \), we have \( U \models \varphi \). Applying the Reflection Principle to \( \varphi \), we find an ordinal \( \beta \) in \( U \) such that \( \varphi \) is absolute between \( V_\beta \) and \( U \); that is, \( V_\beta \models \varphi \). But \( \varphi \leftrightarrow ZF \), so \( V_\beta \models ZF \).

Suppose that \( U \models ZF \) and let \( \beta \) be the least ordinal in \( U \) such that \( V_\beta \models ZF \). Applying Claim with \( V_\beta \) in place of \( U \), we obtain an ordinal \( \gamma < \beta \) and a set \( W \in V_\beta \) with the following properties:

- \( V_\beta \models (W = V_\gamma) \), i.e., \( W \) is “\( V_\beta \)’s version” of \( V_\gamma \);
- \( W \models ZF \).

It remains to show that \( U \models (W = V_\gamma) \). Once this is done, we can conclude that \( V_\gamma \models ZF \) while \( \gamma < \beta \), contradicting the choice of \( \beta \) and hence finishing the proof.

So, for each \( \gamma < \beta \), let \( W_\gamma \in V_\beta \) be “\( V_\beta \)’s version” of \( V_\gamma \). We prove by induction on \( \gamma \) that \( W_\gamma = V_\gamma \).

- For \( \gamma = 0 \), we have \( V_\beta \models (W_0 = \emptyset) \), so \( W_0 = \emptyset = V_0 \).
- For \( \gamma = \delta + 1 \), we have \( V_\beta \models (W_{\delta+1} = P(W_\delta)) \), i.e.,
  \( W_{\delta+1} = P(W_\delta) \cap V_\beta = P(V_\delta) \cap V_\beta = V_{\delta+1} \cap V_\beta = V_{\delta+1} \).
- Finally, if \( \gamma \) is a limit, then
  \[
  x \in W_\gamma \iff x \in V_\beta \land \exists \delta < \gamma (x \in W_\delta) \\
  \iff x \in V_\beta \land \exists \delta < \gamma (x \in V_\delta) \iff x \in V_\beta \cap V_\gamma \iff x \in V_\gamma.
  \]

**Remark.** The above argument can be used to show, more generally, that if \( C \) is a transitive class and \( \gamma \in \text{Ord} \cap C \), then “\( C \)’s version” of \( V_\gamma \) is \( V_\gamma \cap C \).
Problem 7. (20 pts.)

Let $\kappa$ be a cardinal. Recall the following definitions:

- $\kappa$ is **weakly inaccessible** if $\kappa > \aleph_0$, $\text{cf}(\kappa) = \kappa$, and $\kappa > \lambda^+$ for all cardinals $\lambda < \kappa$;
- (AC) $\kappa$ is (strongly) **inaccessible** if $\kappa > \aleph_0$, $\text{cf}(\kappa) = \kappa$, and $\kappa > 2^\lambda$ for all cardinals $\lambda < \kappa$.

Suppose that $\kappa$ is a weakly inaccessible cardinal. Show that

$L \models \text{“} \kappa \text{ is strongly inaccessible} \text{“}.$

First, note that $L \models \text{GCH}$, so

$L \models \text{“} \kappa \text{ is strongly inaccessible} \text{”} \iff \text{“} \kappa \text{ is weakly inaccessible} \text{“}.$

This, we just need to show that $\kappa$ is weakly inaccessible in $L$. To that end, we simply observe that

$“ \kappa \text{ is weakly inaccessible} “$ is a $\Pi_1$-statement, since it is equivalent to

$$
\begin{align*}
\kappa \in \text{Card} & \wedge \kappa > \aleph_0 \wedge \\
\forall \gamma < \kappa \forall f (f \text{ is a function from } \gamma \text{ to } \kappa & \rightarrow f \text{ is not cofinal in } \kappa) \wedge \\
\forall \gamma < \kappa \exists \mu < \kappa (\gamma < \mu & \wedge \mu \in \text{Card}).
\end{align*}
$$
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