COFINAILITY

1. Basics

Example 1.1. Consider the cardinal $\aleph_\omega$. By definition,
$$\aleph_\omega = \sup\{\aleph_n : n < \omega\} = \bigcup\{\aleph_n : n \in \omega\}.$$ 
This means that $\aleph_\omega$ can be expressed as a union of countably many sets of strictly smaller cardinality.

The notion of cofinality is a convenient tool for understanding phenomena of this sort.

Definition 1.2. Let $\alpha$ and $\beta$ be ordinals. A function $f: \alpha \rightarrow \beta$ is called cofinal (in $\beta$) if $\text{ran}(f)$ is unbounded in $\beta$, i.e., if for each $\delta < \beta$ there is some $\gamma < \alpha$ with $f(\gamma) \geq \delta$. The cofinality of an ordinal $\beta$, denoted $\text{cf}(\beta)$, is the least ordinal $\alpha$ such that there is a cofinal function $f: \alpha \rightarrow \beta$.

Exercise 1.3. Let $\beta \in \text{Ord}$. Show that $\text{cf}(\beta)$ is a cardinal and $\text{cf}(\beta) \leq |\beta| \leq \beta$.

Example 1.4. We have
$$\text{cf}(0) = 0, \quad \text{cf}(1) = 1, \quad \text{cf}(100) = 1, \quad \text{cf}(\omega) = \omega, \quad \text{cf}(\omega + 100) = 1, \quad \text{cf}(\omega + \omega) = \omega.$$ 

Example 1.5. As shown in Example 1.1, we have $\text{cf}(\aleph_\omega) = \omega$.

Exercise 1.6. Determine $\text{cf}(\omega^2)$ and $\text{cf}(\varepsilon_0)$ and give examples of cofinal functions
$$\text{cf}(\omega^2) \rightarrow \omega^2 \quad \text{and} \quad \text{cf}(\varepsilon_0) \rightarrow \varepsilon_0.$$ 

Lemma 1.7. Let $\beta$ be an ordinal. Then there is a strictly increasing cofinal function $\text{cf}(\beta) \rightarrow \beta$.

Proof. Let $f: \text{cf}(\beta) \rightarrow \beta$ be an arbitrary cofinal function. Consider the set
$$S := \{\gamma < \text{cf}(\beta) : f(\gamma) > f(\delta) \text{ for all } \delta < \gamma\}.$$ 
The function $f\upharpoonright S: S \rightarrow \beta$ is strictly increasing and cofinal (why?). Since $S$ is well-ordered, there is a strictly increasing bijection $g: \alpha \rightarrow S$ for some ordinal $\alpha$. The existence of a strictly increasing function from $\alpha$ to $\text{cf}(\beta)$ shows that $\alpha \leq \text{cf}(\beta)$; on the other hand, the function $f \circ g: \alpha \rightarrow \beta$ is cofinal, and hence $\alpha \geq \text{cf}(\beta)$. Therefore, $\alpha = \text{cf}(\beta)$ and the function $f \circ g$ is as desired. ■

Lemma 1.8. Let $\alpha$ and $\beta$ be ordinals. If there is a strictly increasing cofinal function $f: \alpha \rightarrow \beta$, then $\text{cf}(\alpha) = \text{cf}(\beta)$.

Proof. It is clear that $\text{cf}(\beta) \leq \text{cf}(\alpha)$ (why?). To establish the other inequality, consider a cofinal function $g: \text{cf}(\beta) \rightarrow \beta$. Define $h: \text{cf}(\beta) \rightarrow \alpha$ as follows:
$$h(\gamma) := \min\{\delta < \alpha : f(\delta) \geq g(\gamma)\}.$$ 
Since the function $f$ is cofinal in $\beta$, given any $\gamma < \text{cf}(\beta)$, there is some $\delta < \alpha$ such that $f(\delta) \geq g(\gamma)$; hence, the function $h$ is well-defined. We claim that $h$ is cofinal in $\alpha$, proving that $\text{cf}(\beta) \geq \text{cf}(\alpha)$. Indeed, consider any $\delta < \alpha$. Since $g$ is cofinal in $\beta$, there is some $\gamma < \text{cf}(\beta)$ with $g(\gamma) \geq f(\delta)$. Then $h(\gamma) \geq \delta$ (why?), and we are done. ■

Exercise 1.9. Show that the operation $\text{cf}$ is idempotent, i.e., for each $\beta \in \text{Ord}$, we have
$$\text{cf}(\text{cf}(\beta)) = \text{cf}(\beta).$$ 

Exercise 1.10. Show that if $\alpha$ is a limit ordinal, then $\text{cf}(\aleph_\alpha) = \text{cf}(\alpha)$. 

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Example 1.11. As observed before, we have \( \text{cf}(\aleph_\omega) = \text{cf}(\omega) = \omega \). Similarly, 
\[
\text{cf}(\aleph_{\omega^2}) = \text{cf}(\aleph_\omega^2) = \text{cf}(\omega^2) = \omega.
\]

2. Regular and inaccessible cardinals

We already know that \( \text{cf}(\aleph_\alpha) = \text{cf}(\alpha) \) if \( \alpha \) is a limit ordinal. What happens if \( \alpha \) is a successor ordinal? For instance, what is \( \text{cf}(\aleph_1) \)? The answer (under \( \text{AC} \)) is given by the following theorem:

**Theorem 2.1** (\( \text{AC} \)). *Let \( \kappa \) be an infinite cardinal. Then \( \text{cf}(\kappa^+) = \kappa^+ \).*

**Proof.** Let \( f : \alpha \to \kappa^+ \) be a cofinal function. Since \( f \) is cofinal, we can write
\[
\kappa^+ = \bigcup \{ f(\beta) : \beta < \alpha \}. \tag{2.2}
\]
For each \( \beta < \alpha \), we have \( f(\beta) < \kappa^+ \), and hence \( |f(\beta)| < \kappa^+ \). Since \( \kappa^+ \) is the least cardinal greater than \( \kappa \), we conclude that \( |f(\beta)| \leq \kappa \). Therefore, (2.2) expresses \( \kappa^+ \) as a union of \( |\alpha| \)-many sets of cardinality at most \( \kappa \), and hence
\[
\kappa^+ \leq |\alpha| \times \kappa = \max\{|\alpha|, \kappa\},
\]
which implies that \( \alpha \geq |\alpha| \geq \kappa^+ \), as desired.

We should perhaps spell out the last step in more detail to clarify how \( \text{AC} \) is used there. In order to show that \( \kappa^+ \leq |\alpha| \times \kappa \), we need to fix a surjection \( g_\beta : \kappa \to f(\beta) \) for each \( \beta < \alpha \), and this is where \( \text{AC} \) gets involved. (Even though the sets \( \kappa \) and \( f(\beta) \) are well-ordered, \emph{a priori} there is no “canonical” choice of a surjection \( \kappa \to f(\beta) \)). Then the map \( g : \alpha \times \kappa \to \kappa^+ \) given by \( g(\beta, \gamma) := g_\beta(\gamma) \) is surjective, and hence \( \kappa^+ \leq |\alpha \times \kappa| \), which is what we wanted to show. \( \blacksquare \)

**Definition 2.3.** An infinite cardinal \( \kappa \) is **regular** if \( \text{cf}(\kappa) = \kappa \).

Thus, Theorem 2.1 asserts that infinite successor cardinals are regular.

**Example 2.4.** Under \( \text{AC} \), we have \( \text{cf}(\aleph_1) = \aleph_1 \), \( \text{cf}(\aleph_{\omega+5}) = \aleph_{\omega+5} \), and \( \text{cf}(\aleph_{\aleph_3}) = \aleph_3 \).

Perhaps somewhat surprisingly, without \( \text{AC} \) Theorem 2.1 may fail, even though it talks about cardinals, which are by definition well-ordered. In fact, under some standard extra assumptions, Gitik showed that, without \( \text{AC} \), it is possible that all infinite cardinals have cofinality \( \omega \! \! \! . \)

**Definition 2.5.** A cardinal \( \kappa \) is called **weakly inaccessible** if \( \kappa > \omega \), \( \kappa \) is regular, and \( \kappa \) is a limit cardinal (i.e., \( \kappa > \lambda^+ \) for all cardinals \( \lambda < \kappa \)). A cardinal \( \kappa \) is called **strongly inaccessible** (or just **inaccessible**) if \( \kappa > \omega \), \( \kappa \) is regular, and \( \kappa > 2^\lambda \) for all cardinals \( \lambda < \kappa \).

Let \( \kappa \) be a weakly inaccessible cardinal. We can write \( \kappa = \aleph_\alpha \) for some (limit) ordinal \( \alpha \). Then
\[
\kappa = \text{cf}(\kappa) = \text{cf}(\aleph_\alpha) = \text{cf}(\alpha) \leq \alpha \leq \kappa,
\]
so \( \kappa = \alpha \). In other words, \( \kappa \) satisfies \( \kappa = \aleph_\kappa \), i.e., \( \kappa \) is a fixed point of the aleph function. This property, however, is not sufficient for weak inaccessibility. For instance, the cardinal
\[
\lambda := \sup\{\aleph_0, \aleph_\aleph_0, \aleph_{\aleph_0}, \ldots\}
\]
satisfies \( \lambda = \aleph_\lambda \), but \( \text{cf}(\lambda) = \omega \), so \( \lambda \) is not regular.

The word “inaccessible” is justified here because an inaccessible cardinal \( \kappa \) cannot be approximated from below by any sequence of smaller cardinals of length less than \( \kappa \). It turns out that inaccessible cardinals must be so “large” that their existence cannot be proven within \( \text{ZFC} \). Also, we will later show that if \( \kappa \) is an inaccessible cardinal, then the set \( V_\kappa \) is so “large” and “rich” that it already is a model of \( \text{ZFC} \! \! \! . \) Inaccessibility is a basic example of what is called a “large cardinal property.” A significant part of modern set theory is done under the assumption that certain “sufficiently large” inaccessible cardinals exist.
3. König’s lemma

**Lemma 3.1** (König). Let $\kappa$ be an infinite cardinal. Then there is no surjection $\kappa \rightarrow \text{cf}(\kappa)_\kappa$. Hence, assuming AC, we have $\kappa^{\text{cf}(\kappa)} > \kappa$.

**Proof.** Suppose, towards a contradiction, that $\kappa \rightarrow \text{cf}(\kappa)_\kappa: \alpha \mapsto f_\alpha$ is a surjection; that is, for each $\alpha < \kappa$, $f_\alpha$ is a function from $\text{cf}(\kappa)$ to $\kappa$ and $\text{cf}(\kappa)_\kappa = \{ f_\alpha : \alpha < \kappa \}$. Also, let $g: \text{cf}(\kappa) \rightarrow \kappa$ be a cofinal function. We define $f: \text{cf}(\kappa) \rightarrow \kappa$ as follows. For each $\gamma < \text{cf}(\kappa)$, consider the set

$$S_\gamma := \{ f_\alpha(\gamma) : \alpha < g(\gamma) \} \subseteq \kappa.$$

By definition, $|S_\gamma| \leq |g(\gamma)| < \kappa$, so the set $\kappa \setminus S_\gamma$ is nonempty. Let $f(\gamma) := \min(\kappa \setminus S_\gamma)$.

Since $f$ is a function from $\text{cf}(\kappa)$ to $\kappa$, there must exist some $\alpha < \kappa$ such that $f = f_\alpha$. Let $\gamma < \text{cf}(\kappa)$ be an arbitrary ordinal such that $g(\gamma) > \alpha$ (such $\gamma$ exists since $g$ is cofinal in $\kappa$ and $\kappa$ is a limit ordinal). Then $f_\alpha(\gamma) \in S_\gamma$, while $f(\gamma) \notin S_\gamma$ by definition; this contradiction completes the proof. ■

**Corollary 3.2** (AC). $2^{\aleph_0} \neq \aleph_\omega$.

**Proof.** If $2^{\aleph_0}$ were equal to $\aleph_\omega$, then we could have $\text{cf}(2^{\aleph_0}) = \text{cf}(\aleph_\omega) = \aleph_0$, while

$$(2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0 \times \aleph_0} = 2^{\aleph_0},$$

contradicting König’s lemma. ■

**Exercise 3.3** (AC). Let $I$ be an arbitrary index set and let $\{ A_i : i \in I \}$ and $\{ B_i : i \in I \}$ be two families of sets indexed by $I$. Let

$$\bigcup \{ A_i : i \in I \} := \bigcup \{ A_i \times \{ i \} : i \in I \}$$

denote the disjoint union of the sets $A_i$, and let

$$\prod \{ B_i : i \in I \} := \{ f : f \text{ is a function with } \text{dom}(f) = I \text{ and } f(i) \in B_i \text{ for all } i \in I \}$$

be the product of the sets $B_i$. Suppose that for each $i \in I$, we have $|A_i| < |B_i|$. Show that

$$\left| \bigcup \{ A_i : i \in I \} \right| < \left| \prod \{ B_i : i \in I \} \right|.$$