BERNSTEIN–CANTOR–SCHRÖDER THEOREM

Here we give a “choiceless” proof of the Bernstein–Cantor–Schröder theorem.

Throughout, we assume the axiom system ZF\(^{-}\). Recall that the notation \(A \lessdot B\) (resp. \(A \approx B\)) means that there is an injection (resp. a bijection) \(A \to B\).

**Theorem** (Bernstein–Cantor–Schröder). Let \(A\) and \(B\) be sets. If \(A \lessdot B\) and \(B \lessdot A\), then \(A \approx B\).

Furthermore, if \(f : A \to B\) and \(g : B \to A\) are injective functions, then there is a bijection \(h : A \to B\) such that \(h \lessdot f \cup g^{-1}\) (i.e., for each \(a \in A\), \(h(a) = f(a)\) or \(h(a) = g^{-1}(a)\)).

**Proof.** Let \(f : A \to B\) and \(g : B \to A\) be injections. Recursively define sets \(A_n\) and \(B_n\), for each \(n \in \omega\), as follows:

\[
B_0 := B \setminus f[A]; \quad A_n := g[B_n]; \quad B_{n+1} := f[A_n].
\]

Let

\[
A' := \bigcup \{A_n : n \in \omega\} \quad \text{and} \quad A'' := A \setminus A';
\]

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\]

Note that \(A' \lessdot \text{ran}(g)\), so we can define a function \(h : A \to B\) by

\[
h(a) := \begin{cases}
  f(a) & \text{if } a \in A''; \\
  g^{-1}(a) & \text{if } a \in A'.
\end{cases}
\] (*)&

It is clear that \(h \lessdot f \cup g^{-1}\), so it remains to show that \(h\) is a bijection.

**Claim.** If \(a \in A'\), then \(h(a) \in B'\), while if \(a \in A''\), then \(h(a) \in B''\).

**Proof.** Suppose that \(a \in A'\) and let \(n \in \omega\) be such that \(a \in A_n\). Since \(A_n = g[B_n]\), we have

\[
h(a) = g^{-1}(a) \in B_n \subseteq B',
\]

as claimed. The proof in the case when \(a \in A''\) is left as an exercise. \(\dagger\)

**Claim.** The function \(h\) is injective.

**Proof.** Let \(a_0, a_1 \in A\) be elements such that \(h(a_0) = h(a_1)\). From the previous claim it follows that either \(a_0\) and \(a_1\) both belong to \(A'\), or else, they both belong to \(A''\). If \(a_0, a_1 \in A'\), then \(g^{-1}(a_0) = g^{-1}(a_1)\), which implies that \(a_0 = a_1\), as desired. On the other hand, if \(a_0, a_1 \in A''\), then \(f(a_0) = f(a_1)\), and hence \(a_0 = a_1\) again due to the injectivity of \(f\). \(\dagger\)

It remains to show that \(h\) is surjective. To that end, take any \(b \in B\). We wish to find an element \(a \in A\) with \(h(a) = B\). If \(b \in B'\), then \(b = h(g(b))\), and we are done. On the other hand, if \(b \in B''\), then \(b = h(f^{-1}(b))\) (why?), and we are done again. \(\blacksquare\)

In the remainder of this note we will try to explain how one can come up with a formula such as (*). The insight is to treat just one element of the set \(A \cup B\) at a time. For convenience, assume that the sets \(A\) and \(B\) are disjoint. Given an element \(x \in A \cup B\), we can consider its orbit \(O_x\), i.e., the set of all those elements of \(A \cup B\) that can be obtained from \(x\) by repeatedly applying the functions \(f, g, f^{-1},\) and \(g^{-1}\) finitely many times. For instance, if \(a \in A\), the orbit \(O_a\) can look like this:

\[
\ldots \xrightarrow{g} f^{-1}(g^{-1}(a)) \xrightarrow{f} g^{-1}(a) \xrightarrow{g} a \xrightarrow{f} f(a) \xrightarrow{g} g(f(a)) \xrightarrow{f} \ldots
\]
In general, since the functions \( f \) and \( g \) are injective, it is not hard to see that there are only four possibilities for the general “shape” of such an orbit:

**Type I:** A finite cycle.

**Type II:** A path infinite in both directions.

**Type IIIa:** An infinite path starting in \( A \setminus g[B] \).

**Type IIIb:** An infinite path starting in \( B \setminus f[A] \).

Notice also that for any two elements \( x, y \in A \cup B \), the orbits \( O_x \) and \( O_y \) are either disjoint or equal. In other words, the set \( A \cup B \) is partitioned into pairwise disjoint orbits.

Imagine for a moment that there are no orbits of type IIIb. Then we can simply set \( h = f \), since the function \( f \) is clearly a bijection on the orbits of types I, II, and IIIa:

**Type I.**

**Type II.**

**Type IIIa.**

Unfortunately, on the orbits of type IIIb the function \( f \) fails to be surjective:
Thanksfully, on the orbits of type IIIb, the function \( g \) is a bijection:

Therefore, for each \( a \in A \), we can make \( h(a) \) equal to \( f(a) \) if the orbit of \( a \) is of type I, II, or IIIa, while if the orbit of \( a \) is of type IIIb, then we can set \( h(a) := g^{-1}(a) \). All that is left to observe is that the union of all the orbits of type IIIb is precisely the set \( A' \cup B' \) as defined in the above proof of the Bernstein–Cantor–Schröder theorem, since an element belongs to an orbit of type IIIb if and only if it can be obtained by repeatedly applying the functions \( f \) and \( g \) to an element in \( B \setminus f[A] \):

\[
\begin{align*}
\bullet & \xrightarrow{g} \bullet \xrightarrow{f} \bullet \xrightarrow{g} \bullet \xrightarrow{f} \bullet \xrightarrow{g} \ldots \\
& \in B \setminus f[A] = B_0 \\
& \in g[B_0] = A_0 \\
& \in f[A_0] = B_1 \\
& \in g[B_1] = A_1 \\
& \in f[A_1] = B_2 \\
& \ldots
\end{align*}
\]