Problem 1 (Ranks of groups). The rank of a group $G$, in symbols $\text{rank}(G)$, is the smallest cardinality of a set $S \subseteq G$ such that $G = \langle S \rangle$. For instance:

- $\text{rank}(G) = 0$ if and only if $G$ is trivial;
- $\text{rank}(G) \leq 1$ if and only if $G$ is cyclic.

(a) (Warm-up; not for credit.) Show that every subgroup of a cyclic group is cyclic. In other words, if $G$ is a group of rank at most 1 and $H \leq G$, then $\text{rank}(H) \leq 1$ as well.

The goal of this problem is to show that, in contrast to the statement in part (a), a group of rank 2 can have a subgroup of infinite rank!

Let $g, h \in \text{Sym}(\mathbb{R})$ denote the functions given by

$$g(x) := x + 1 \quad \text{and} \quad h(x) := x/2 \quad \text{for all } x \in \mathbb{R}.$$  

Let $G := \langle g, h \rangle$ be the subgroup of $\text{Sym}(\mathbb{R})$ generated by $g$ and $h$.

(b) Show that the rank of $G$ is 2.

(c) Show that for all $n, m \in \mathbb{Z}$, the function $f_{n,m} \in \text{Sym}(\mathbb{R})$ is in $G$, where

$$f_{n,m}(x) := x + \frac{n}{2^m} \quad \text{for all } x \in \mathbb{R}.$$  

(d) Show that $H := \{f_{n,m} : n, m \in \mathbb{Z}\}$ is a subgroup of $G$.

(e) Show that the rank of $H$ is infinite.

Problem 2. Given $n \in \mathbb{N}$, let $F_n$ denote the free group with $n$ generators; that is, $F_n = F_{\{x_1, \ldots, x_n\}}$, where $x_1, \ldots, x_n$ are pairwise distinct elements. Similarly, let $F_{\infty}$ denote the free group generated by a countably infinite set of generators: $F_\infty = F_{\{x_1, x_2, x_3, \ldots\}}$. It is immediate from the definition that $\text{rank}(F_n) \leq n$ for all $n \in \mathbb{N}$.

(a) (Warm-up; not for credit.) Show that $\text{rank}(F_0) = 0$, $\text{rank}(F_1) = 1$, and $\text{rank}(F_2) = 2$.

We will later show that in fact $\text{rank}(F_n) = n$ for all $n \in \mathbb{N}$, while $\text{rank}(F_\infty)$ is infinite. This might seem like an obvious observation, but in fact the proof is a bit subtle. I encourage you to attempt it as a challenge! Even the following is somewhat tricky:

(b) (Bonus; not for credit.) Show that the groups $F_2$ and $F_3$ are not isomorphic.

To illustrate the difficulties that arise here, consider the free group $F_2 = F_{\{x, y\}}$ with generators $x$ and $y$ and let $H \leq F_2$ be the subgroup defined by $H := \langle x^n y x^{-n} : n \in \mathbb{N} \rangle$.

(c) Show that $H \cong F_\infty$.

In other words, $F_2$ has a subgroup isomorphic to $F_\infty$!

Problem 3 (Products and coproducts of Abelian groups). In this problem we work in the category $\text{Ab}$ of Abelian groups. Suppose that we are given a family of Abelian groups $(G_i)_{i \in I}$ indexed by the elements of some set $I$ (such as $I = \{1, \ldots, n\}$ or $I = \mathbb{N}$). The product of $(G_i)_{i \in I}$ is an Abelian group $\prod_{i \in I} G_i$ together with homomorphisms $p_i: \prod_{i \in I} G_i \to G_i$ satisfying the following
universal property: Whenever $H$ is an Abelian group and $q_i : H \to G_i$ are homomorphisms, there exists a unique homomorphism $\varphi : H \to \prod_{i \in I} G_i$ such that $q_i = p_i \circ \varphi$ for all $i \in I$:

Similarly, the coproduct of $(G_i)_{i \in I}$, also called the direct sum of $(G_i)_{i \in I}$, is an Abelian group $\bigoplus_{i \in I} G_i$ together with homomorphisms $f_i : G_i \to \bigoplus_{i \in I} G_i$ satisfying the following universal property: Whenever $H$ is an Abelian group and $h_i : G_i \to H$ are homomorphisms, there exists a unique homomorphism $\varphi : \bigoplus_{i \in I} G_i \to H$ such that $h_i = \varphi \circ f_i$ for all $i \in I$:

Provide an explicit construction of the groups $\prod_{i \in I} G_i$ and $\bigoplus_{i \in I} G_i$.

Hint: When $I$ is finite, the groups $\prod_{i \in I} G_i$ and $\bigoplus_{i \in I} G_i$ coincide, but they differ when $I$ is infinite.

**Problem 4** (Internal direct products). Let $G$ be a group and let $H_1, H_2 \leq G$ be subgroups. Consider the following function $\varphi : H_1 \times H_2 \to G$:

$$\varphi(h_1, h_2) := h_1 h_2 \quad \text{for all } h_1 \in H_1 \text{ and } h_2 \in H_2.$$ 

We say that $G$ is the internal direct product of $H_1$ and $H_2$ if the map $\varphi : H_1 \times H_2 \to G$ is a group isomorphism. Show that $G$ is the internal direct product of $H_1$ and $H_2$ if and only if the following conditions are satisfied:

- both $H_1$ and $H_2$ are normal in $G$;
- $H_1 \cap H_2 = \{e\}$, where $e$ is the identity element of $G$;
- $\langle H_1 \cup H_2 \rangle = G$.

**Problem 5.** Show that the dihedral group $D_{12}$ is isomorphic to $\text{Sym}\{1, 2, 3\} \times \text{Sym}\{1, 2\}$.

**Problem 6.** Consider the group $Q := \langle x, y \mid x^4, x^2 y^{-2}, xy^{-1} xy \rangle$.

(a) Show that $|Q| = 8$.

(b) Show that $Q$ is not isomorphic to the dihedral group $D_8$ (even though $|D_8| = 8$).